DOCUMENT RESUME

ED 173 094 SE 027 901

TITLE	School Mathematics Study Group, Unit Number Twelve. Chapter 19 - Rigid Motions and Coordinates. Chapter 20 - Squares and Rectangles. Chapter 21 - Square Roots and Real Numbers.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	68
NOTE	94p.; Not available in hard copy due to small, light and broken type
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Algebra; *Analytic Geometry; Curriculum; *Geometry; *Instruction; Mathematics Education; *Number Concepts; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is unit twelve of a fifteen-unit SMSG secondary school text for high school students. The text is devoted almost entirely to mathematical concepts which all citizens should know in order to function satisfactorily in our society. Chapter topics include rigid motions and coordinates, squares and rectangles, and square roots and real numbers. (MP)

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UNIT NUMBER TWELVE

Chapter 19. Rigid Motions and Coordinates

Chapter 20. Squares and Rectangles

Chapter 21. Square Roots and Real Numbers

SE 027 901

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Chapter 19

RIGID MOTIONS AND COORDINATES

19-1. Rigid Motions in a Plane

Figure B was obtained from Figure A as follows:

A copy of Figure A was made on a tracing sheet which was then moved to the position where Figure B could be drawn as shown.

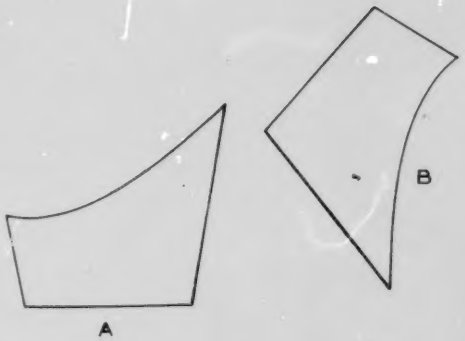


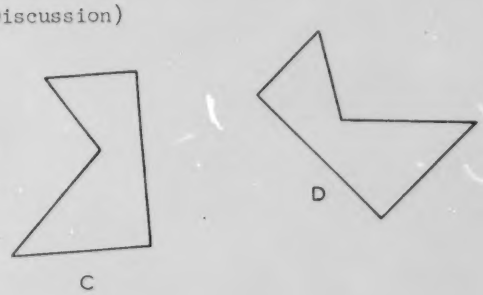
Figure B is called an image of Figure A. Since Figure B is a duplicate of Figure A, the two figures are congruent.

You recall that any motion like this in a given plane which results in the image being congruent to the original figure is called a rigid motion in the plane.

Exercises 19-1a

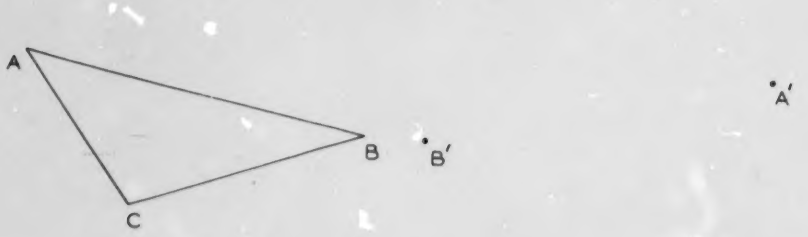
(Class Discussion)

1. Consider Figure C and its image Figure D. Make a tracing of Figure C and move the tracing sheet to the position where the tracing coincides with Figure D. Did you have any difficulty?



From this example we see that for some rigid motions, the image of a figure is obtained by turning over the tracing sheet.

2. Copy the picture below.



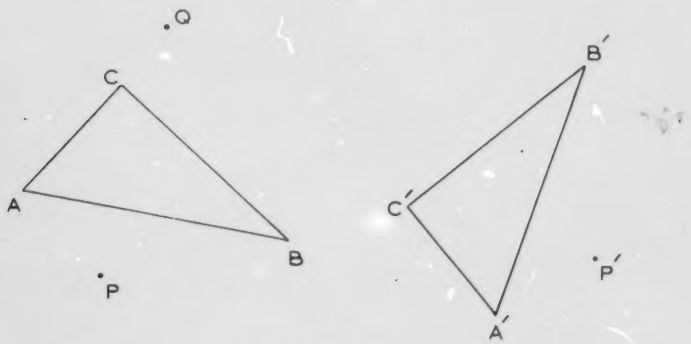
In a rigid motion of triangle ABC, the image of A is A' and the image of B is B' as shown above. There are two possible locations for the image of C, which can be found as follows:

(a) Make a copy of △ ABC on a tracing sheet. Move the tracing sheet <u>without</u> flipping it over so that A and B coincide respectively with A' and B'. How is C' the image of C determined? Why must △ ABC and △ A'B'C' be congruent? Draw △ A'B'C' the image of △ ABC.

(b) Now flip the tracing sheet over so that A and B coincide respectively with A' and B'; mark the location of C" the image of C for this motion. Draw △ A'B'C". which is another image of △ ABC.

From this example, we see that for some rigid motions, it is necessary to show the images of at least three non-collinear points before the rigid motion is determined.

3. Given a rigid motion which when applied to △ABC results in the image △ A'B'C'.



(a) Make a tracing of $\triangle ABC$ and points P and Q. Move the tracing sheet so that the copy of $\triangle ABC$ coincides with $\triangle A'B'C'$. Was it necessary to flip over the tracing sheet? Notice that the same movement of the tracing sheet which brings the copy of $\triangle ABC$ to coincide with $\triangle A'B'C'$ also brings the copy of point P to point P'. This means that for this rigid motion, P' is the image of P.

(b) Mark the location of Q', the image of Q for the given rigid motion.

(c) Mark a point R that lies on $\triangle ABC$ and find its image R' for this rigid motion. Does R' lie on $\triangle A'B'C'$?

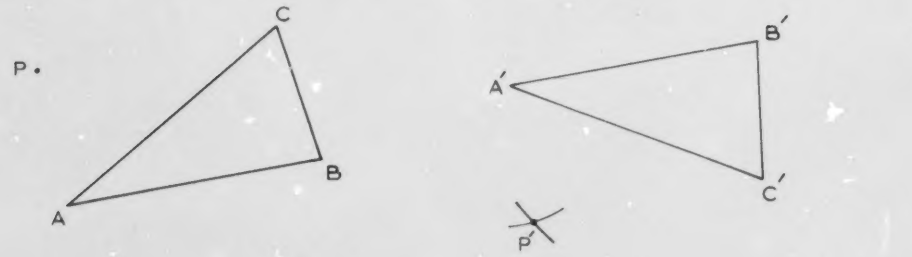
(d) Mark any point S. Now apply the given rigid motion to S and show the resulting image S'. This example shows that a rigid motion in a plane determined by a triangle and its image applies to all points and figures in the plane of the given triangle as well.

4. Now study the drawing for Exercise 3 above and compare some distances for the given rigid motion.

(a) Does the distance PP' equal the distance QQ'? Are the distances AA', BB', and CC' the same? What is your conclusion? For any given rigid motion are the distances between points and their images necessarily the same?

(b) Now check the distances PQ and P'Q'. Are they the same? Actually you would know in advance that these distances are the same because $\overline{PQ}$ and $\overline{P'Q'}$ must be congruent. Why must they be congruent? Does PA equal P'A' and PB equal P'B'? Does QC equal Q'C'? Check the distances between other pairs of points and between their corresponding images.

5. Given: A rigid motion determined by $\triangle ABC$ and its image $\triangle A'B'C'$.



Point P', the image of P, can be found without using a tracing sheet as follows:

With AP as radius and A' as center, draw an arc as shown above.

With CP as radius and C' as center, draw another arc intersecting the first arc.

Explain why the intersection of the arcs is the image point P'.

From the exercises above we see that if the images of three non-collinear points are known for any rigid motion, then the image of any other point in the plane can be found for the same rigid motion.

Exercises 19-1b

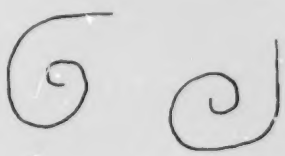
Note: (For the following exercises, make a copy of the diagrams when necessary and do the work on your paper instead of in the text.)

1. For each rigid motion in which Figure II is the image of Figure I, tell whether or not it was necessary to flip over the tracing sheet to obtain the image.

(a)

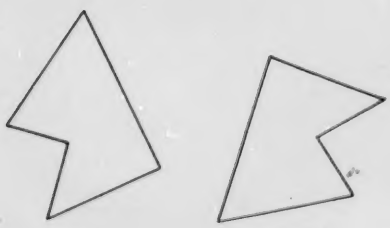
I II

(b)



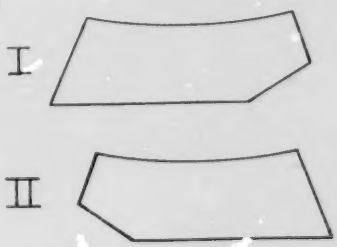
I II

(c)



I II

(d)



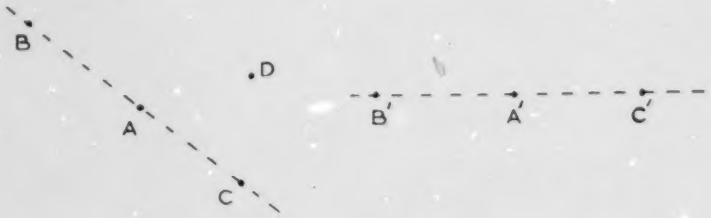
I II

2. For all the rigid motions in which A' is the image of A, what are the possible positions for B', the image of B in the plane?

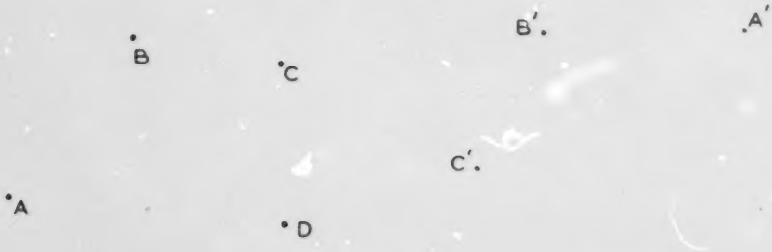
3. For all the rigid motions in which A' is the image of A and B' is the image of B, what are the possible positions for C', the image of C in the plane?

4. For all the rigid motions in which A', B', and C' are respectively the images of A, B, and C, what are the possible positions of D', the image of D in the plane?

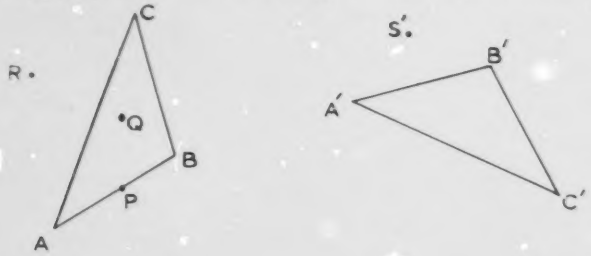
(a)



(b)



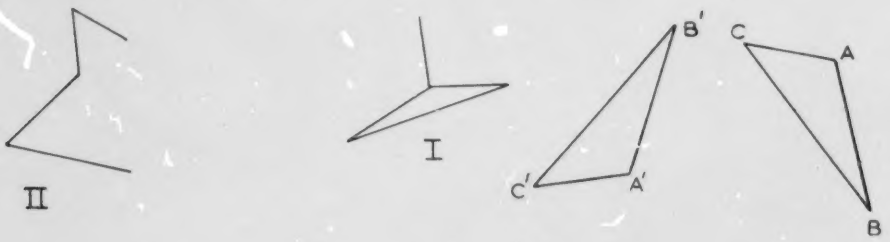
5. Given: A rigid motion determined by $\triangle ABC$ and its image $\triangle A'B'C'$.



(a) Find the images of P, Q, and R.

(b) Find the point S whose image is S'.

6. Given: A rigid motion determined by $\triangle ABC$ and its image $\triangle A'B'C'$.



Find the image of Figure I and of Figure II.

7. Given: A rigid motion determined by points A, B, and C and their images A', B', and C'.

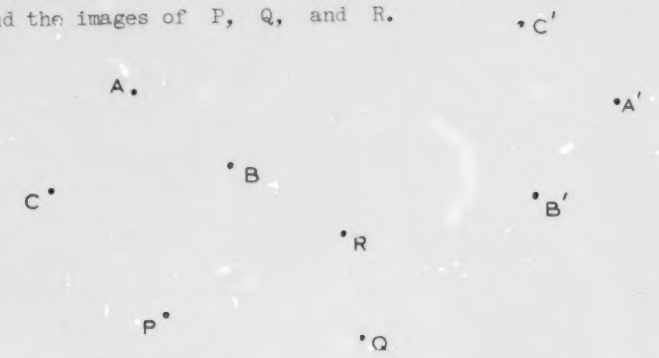
Without using a tracing sheet,

(a) find the images of P, Q, R, and S;

(b) find the point T whose image is T'.

8. Explain why points A, B, C and their images A', B', C' do not determine a rigid motion.

9. In a rigid motion determined below by A, B, C and A', B', C', find the images of P, Q, and R.



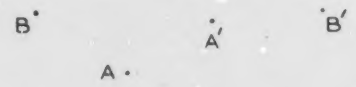
Which of the points is fixed in this rigid motion?

19-2. Properties of Rigid Motion

In this study of motion, we are not interested in the paths but only in the results of motions. When a motion is applied to a figure the result is an image, and if the image and figure are congruent, then the motion is a rigid motion.

The use of a tracing sheet helps to show that a rigid motion not only assigns a particular image to a particular figure but also assigns to each point in the plane a unique image point in the plane.

This assignment of unique images to points is called a function or a mapping. If A and B are any points in a plane, and A' and B' are their images for a given rigid motion, then the distances AB and A'B' are equal.



This is what we mean when we say that a rigid motion preserves distances.

Definition. A rigid motion is a function or a mapping of a plane onto itself which preserves distances.

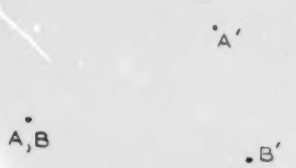
We shall use the notation, $A \rightarrow A'$, to indicate that A' is the image of A.

The rigid motion shown in Exercise 9 in the previous section has a fixed point. In other words, point Q and its image Q' are the same point. We say that this point is invariant for the rigid motion. Invariant points will play an important part in our study of rigid motions.

Exercises 19-2a

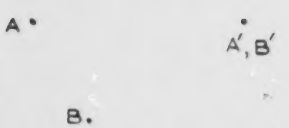
(Class Discussion)

1. Since a rigid motion has been defined as a function, a point cannot be assigned two different images. There is also another reason why a point cannot have two distinct images in a rigid motion.



Let's suppose that A and B are the same point but A' and B' are Different images in a rigid motion. Are the distances AB and A'B' the same? Why? Then what contradiction has been reached?

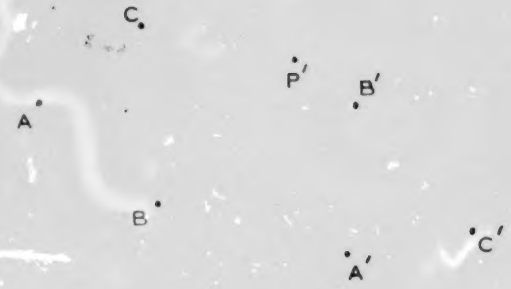
2. In a rigid motion can two different points have the same image?



Let's suppose that A and B are different points but their images A' and B' are the same point in a rigid motion. Show that this leads to a contradiction.

3. We have already seen that in a rigid motion, each point is assigned an image point which may or may not be the point itself. Now let's turn it around. Is each point in the plane the image of some point? In the rigid motion determined on the following page, by A, B, C and A', B', C', let P' be any point on the plane. We shall find the point P which has P' as its image as follows.

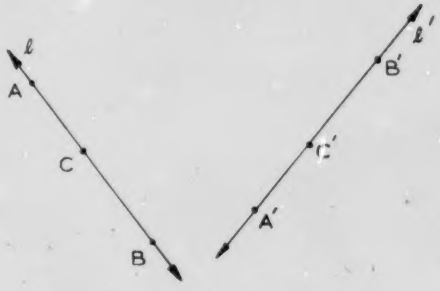
(a) Make a tracing of points A, B, and C, and move the sheet so that the tracing coincides with the image points A', B', and C'. Is it necessary to flip over the tracing sheet?



(b) Mark point P' while the tracing coincides with points A', B', C'. Now show how you would find point P. This example illustrates the fact that for a rigid motion each point in the plane is the image of some point and the result is a one-to-one correspondence of the points in the plane.

4. What is the image of a line in a rigid motion? Suppose that we are given a line ℓ with two points A and B on ℓ and their images A' and B' in a rigid motion.

(a) Let ℓ' be the line determined by A' and B'. We must prove that ℓ' is the image of ℓ. Let C be any point on ℓ between A and B. What must we prove about C', the image of C?



(b) Since C' is on ℓ between A and B, then AB = AC + BC. Therefore A'B' = A'C' + B'C'. Why? Then C' is on ℓ' between A' and B'.

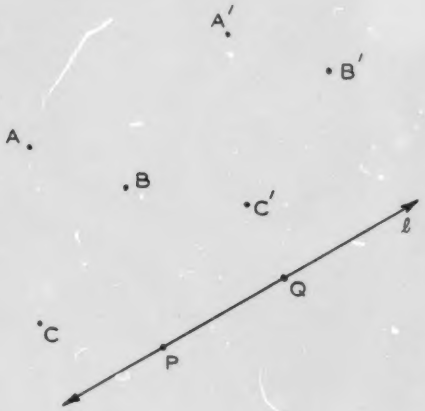
(c) There are two other cases for the location of point C on ℓ. One possibility is that C may be chosen so that B is between A and C. What is the other possibility for choosing C? Prove that in both cases C' lies on ℓ'.

(d) Then ℓ' is the image of ℓ. Why?

5. Given: The line ℓ and the rigid motion determined by A, B, C and A', B', C'.

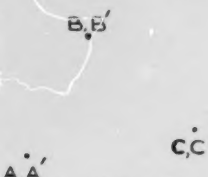
(a) Find the images of points P and Q on ℓ.

(b) Choose any other point R on ℓ and find its image. Since the image of any point on ℓ also lies on ℓ, then ℓ is its own image. In other words, line ℓ is invariant for this rigid motion.



Note that although ℓ is invariant for this rigid motion, the points on ℓ are not invariant. Explain this.

6. In the rigid motion determined below, by A, B, C and A', B', C', each of the points A, B, and C, is its own image. Let P be any other point in the plane. Find the image of point P.



Since in this case each point in the plane is a fixed point, you may wonder if we should call this a motion. We shall find it useful to call this function a rigid motion and we shall give it the special name <u>identity motion</u>.

<u>Exercises 19-2b</u>

1. Use the definition of a rigid motion and the fact that the image of a line is a line to prove the following for any rigid motion.

(a) The image of a segment $\overline{AB}$ is a congruent segment $\overline{A'B'}$.

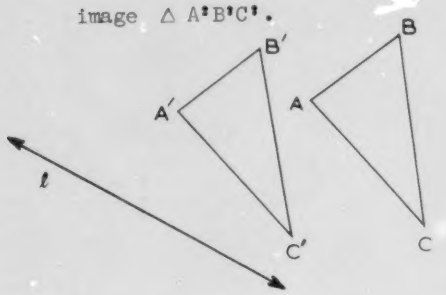
(b) The image of a ray $\overrightarrow{AB}$ is a ray $\overrightarrow{A'B'}$.

2. Use Exercise 1 to prove that for any rigid motion the image of a triangle ABC is the congruent triangle A'B'C'.

3. Use Exercise 2 to prove that the image of an angle ABC is the congruent angle A'B'C'.

4. Prove that the image of a circle is a circle for any rigid motion. Use the definition of a rigid motion.

5. Given: A line ℓ and a rigid motion determined by $\triangle ABC$ and its image $\triangle A'B'C'$.



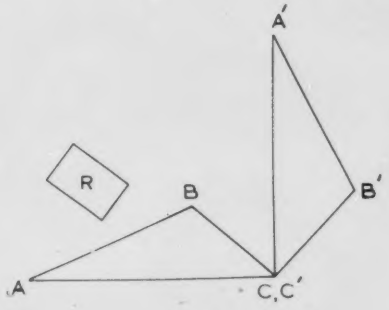
Copy this drawing and find for this rigid motion:

(a) the image of $\triangle A'B'C'$;

(b) the triangle whose image is $\triangle ABC$;

(c) the image of line ℓ.

6. Given: A rigid motion determined by $\triangle ABC$ and its image $\triangle A'B'C'$, and the rectangle as indicated. Copy this drawing and find for this rigid motion:



(a) the image of $\triangle A'B'C'$;

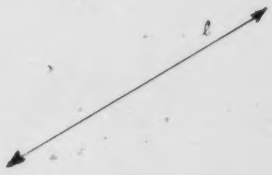
(b) the triangle whose image is $\triangle ABC$;

(c) the image of the rectangle;

(d) the rectangle whose image is the rectangle shown.

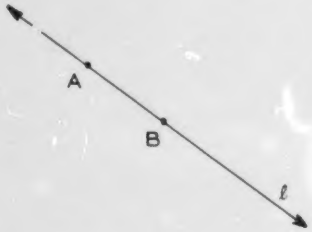
7. Given: A point P. Show a rigid motion with three non-collinear points and their images in which P is a fixed point. Can you think of more than one?

8. Given: A line ℓ. Show a rigid motion with three non-collinear points and their images in which line ℓ is invariant. Can you think of more than one?



9. List the properties of rigid motions that have been developed in the exercises in this section.

10. Given: Line ℓ and a rigid motion which leaves points A and B fixed.



Prove that every point on line ℓ is fixed for this rigid motion.

19-3. Translations

You have already studied some different kinds of rigid motions known as slides, turns, and flips. These special motions are important because they can be used singly or in combination to describe any rigid motion.

In this section we shall take a close look at the rigid motions which were introduced to you earlier as slides. The rigid motion shown below by $\triangle ABC$ and its image $\triangle A'B'C'$ is an example.

The dotted lines connecting points A, B, C with their images A', B', C' show an important pattern. The distances AA', BB', and CC'

are equal, and the directions from the points to their corresponding images are the same.

Now use a tracing sheet and find the image P' of point P for this rigid motion. Show that the distance PP' and the direction from P to P' are the same as for the points A, B, C and their images. Notice that in moving the tracing sheet you do not flip it over.

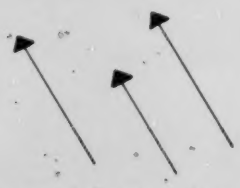
The physical movement of the tracing of $\triangle ABC$ to the position of $\triangle A'B'C'$ is called a <u>slide</u>. However, the one-to-one correspondence of points of the plane and their images resulting from this motion is called a <u>translation</u>.

> <u>Definition</u>. A translation is a rigid motion in which the distances between the points and their images are equal and the directions from the points to their images are the same.

Since in any particular translation the distances and directions from the points to their images are the same for all points, we can determine that particular translation from a single point and its image.

We find it convenient to designate a translation with an arrow. The length of the arrow fixes the distance between each point and its image. The arrow shows the direction from each point to its image.

Be sure not to confuse an arrow with a ray. A ray has direction but no length because a ray has no ending.



Arrows that have the same length and the same direction designate the same translation. Therefore each arrow shown at the left determines the same translation.

The special case where the distance between each point and its image is zero can be thought of as a translation shown by an arrow of zero length. Therefore we can consider the identity motion as a translation.

Exercises 19-3a

(Class Discussion)

1. Given: A translation determined by the arrow shown. Copy the drawing and apply the translation to point P. You can find its image P' by carrying out the following steps.

(a) Draw a line through P parallel to the arrow.

(b) Now explain how you would find P' on the line you drew.

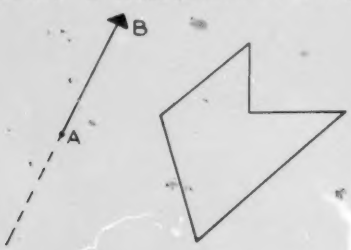
2. Given: A rigid motion in which the distances and directions from points P and Q to their images P' and Q' are the same. To show that this rigid motion is not necessarily a translation:

(a) Mark points P, Q, R on a tracing sheet. Flip the sheet over and place the marks for P and Q on P' and Q'. Now mark the location of R', the image of R. Does the distance RR' equal PP' or QQ'? Is this rigid motion a translation? Explain your answer.

(b) Now use the tracing sheet again to find R' but this time do not flip over the tracing sheet. Does RR' equal PP' or QQ'? Is this rigid motion a translation? Explain your answer.

This example shows how important it is that the tracing sheet not be turned over when used for translations.

3. Given: A translation shown by the arrow. Copy the drawing and find the image of the polygon as follows:



(a). Mark the tail point A of the arrow and the vertices of the polygon on a tracing sheet. With a straightedge draw a tracing of $\overleftrightarrow{AB}$ so that it covers the arrow

and extends as shown by the dotted line.

(b) Slide the tracing sheet so that the mark for A coincides with B, and the tracing of $\overleftrightarrow{AB}$ covers the arrow. $\overleftrightarrow{AB}$ serves as a guide line. Mark the location of the image vertices on your original paper and draw the image polygon.

(c) Why is it necessary to draw an extended tracing of the line $\overleftrightarrow{AB}$?

(d) Describe another way of finding the image of the polygon without using a tracing sheet.

Exercises 19-3b

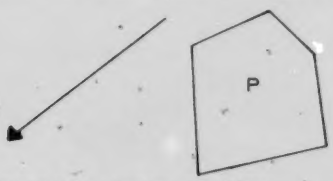
(Note: When necessary, make copies of the drawings.)

1. For a given translation, the image of point A is A'. We indicate this relationship thus: $A \rightarrow A'$.

(a) Find the images of B and C.

(b) Find the points which have B and C as images.

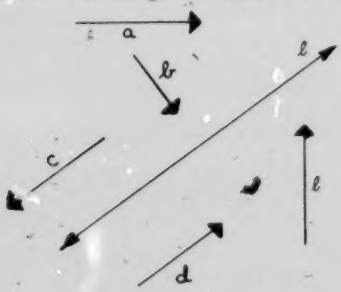
2. For the translation shown by the arrow



(a) Find the image of polygon P.

(b) Find the polygon which has as image polygon P.

3. Find the image of line ℓ for the translation defined by



(a) arrow a

(b) arrow b

(c) arrow c

(d) arrow d

(e) arrow e

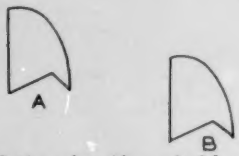
4. Draw an arrow for a translation in which the image of line ℓ is

(a) the line ℓ itself.

(b) a line parallel to ℓ.

(c) a line that intersects ℓ in exactly one point.

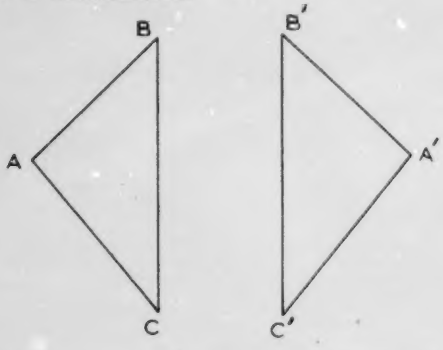
5. Draw an arrow for the translation in which



(a) Figure B is the image of Figure A.

(b) Figure A is the image of Figure B.

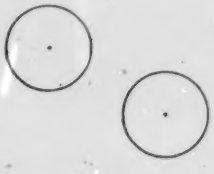
6. Explain why the rigid motion defined by $\triangle ABC$ and its image $\triangle A'B'C'$ is not a translation.



7. For which pairs of figures is there a translation in which the second figure is the image of the first? If there is, show the translation with an arrow.

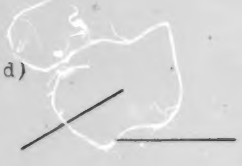
(a)

(b)

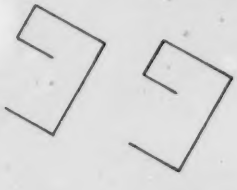


(c)

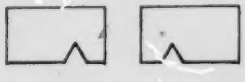
(d)



(e)



(f)



8. Is there a translation other than the identity motion in which there is

(a) a fixed point?

(b) a fixed segment?

(c) a fixed line?

9. For any non-identity translation, describe in words and draw a figure to illustrate the image of

(a) a line

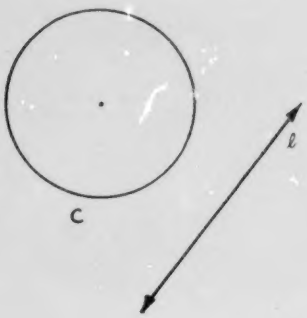
(b) a segment

(c) a ray

(d) a circle

(e) an angle

10.



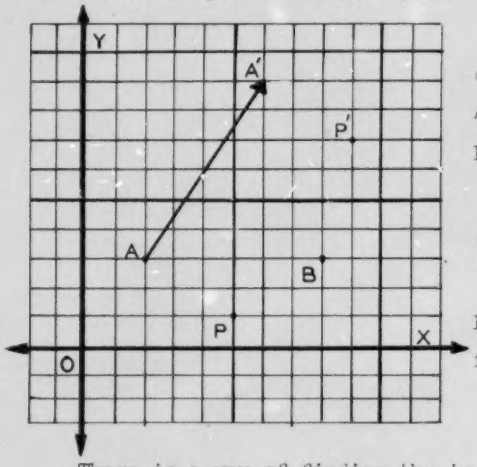
(a) Show with an arrow a translation in which the image of the circle is tangent to line ℓ.

(b) Indicate two other translations in which the image of the circle is tangent to line ℓ.

(c) Indicate two translations in which the image of ℓ is tangent to the circle.

19-4. Translations and Coordinates

In this section we shall see how a translation can be conveniently described by using a coordinate system.



The arrow shown at the left defines a translation. The tail point A is at (2,3) and the arrowhead point A' is at (6,9).

A point P has been located at (5,1).

Using a tracing sheet, the image P' of P is found to be at (9,7) for this translation.

There is a way of finding the image P' of P without using a tracing sheet by taking advantage of the coordinate system.

Notice below that the coordinates of A' (6,9) can be found by adding 4 to the first coordinate of A and by adding 6 to the second coordinate of A.

$A(2,3) \rightarrow A'(2 + 4,\ 3 + 6)$

This same method may be used to find the coordinates of P'(9,7) as is shown at the left.

$P(5,1) \rightarrow P'(5 + 4,\ 1 + 6)$

Using this method to find the image of B(8,3), the result is B'(12,9) as shown at the left.

$B(8,3) \rightarrow B'(8 + 4,\ 3 + 6)$

If B' is plotted on the graph above, its location can be verified by using a tracing sheet.

In general for this translation the image of any point (x,y) is the point $(x + 4,\ y + 6)$.

If we call this translation $\underline{t}$, we can describe the translation as a function thus:

$$t : (x,y) \rightarrow (x + 4,\ y + 6).$$

In a coordinate system, then, a translation can be compactly described as a function, as above.

Exercises 19-4a

(Class Discussion)

1. A translation g is defined by the following function:

$$g : (x,y) \rightarrow (x - 2, y + 1).$$

(a) For any given point, state in words how the coordinates of its image are found.

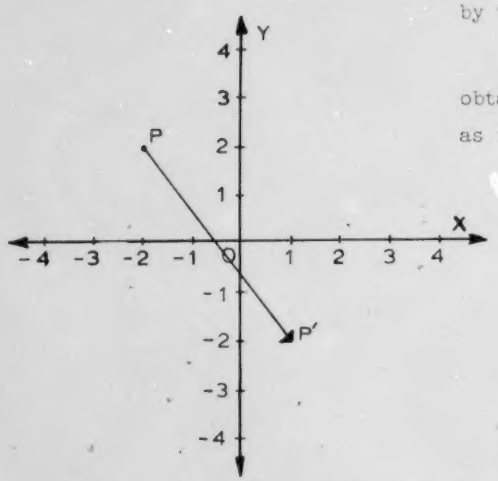
(b) Find the images of the following points for this translation.

A(4,3) B(0,-1) C(0,0) D(2,1)

(c) Using a coordinate system, draw an arrow which defines the translation g.

(d) The image of point E is E'(1,3). Find the coordinates of point E.

2. The image of P(-2,2) is P'(1,-2) for a translation t as shown by the arrow.



The coordinates of P' can be obtained from the coordinates of P as follows:

$$P(-2,2) \rightarrow P'(-2 + 3, 2 - 4).$$

(a) State in words how the coordinates of P' can be found from the coordinates of P.

(b) Now write the function for this translation t.

$t : (x,y) \rightarrow ?$

(c) For this translation, find the images for the following points.

A(7,3), B(-4,-4), C(-3,4), D(0,0).

3. A translation h is defined by the following function:

$$h : (x,y) \rightarrow (x + 4, y).$$

(a) Show that the image of A(2,3) is A'(6,3).

(b) Using a coordinate system, plot the following points and show their images for this translation.

B(-5,-2) C(-2,4) D(1,-3)

(c) Explain how translation h may be described as a translation 4 units to the right.

Exercises 19-4b

1. A translation t is defined by the following function:

$$t : (x,y) \rightarrow (x - 3, y + 2).$$

Find the images of the following points under t.

A(5,-4) B(2,-2) C(0,0) D(3,-2) E(-2,-3)

2. Find the images of the following points in the plane under the translations described by the functions.

(a) (3,-2) by $f : (x,y) \rightarrow (x - 2, y + 3)$

(b) (0,0) by $g : (x,y) \rightarrow (x + 5, y - 4)$

(c) (-2,-4) by $h : (x,y) \rightarrow (x - 1, y + 1)$

(d) (-5,1) by $j : (x,y) \rightarrow (x + 2, y - 3)$

(e) (6,-2) by $k : (x,y) \rightarrow (x, y + 1)$

3. A translation s is given as the following function:

$$s : (x,y) \rightarrow (x + 4, y - 5).$$

Find points A, B, C, D, and E whose images are as follows.

$A'(3,2)$ $B'(-1,-1)$ $C'(0,0)$ $D'(4,-5)$ $E'(-2,1)$

4. In each case, a point and its image under a translation are given. Find the function that describes the translation.

	Point	Image
(a)	A(1,5)	$A'(3,6)$
(b)	B(4,2)	$B'(3,4)$
(c)	C(-1,0)	$C'(2,5)$
(d)	D(3,2)	$D'(0,0)$
(e)	E(0,-4)	$E'(2,-1)$

4.

	Point	Image
(f)	F(2,3)	F'(2,-3)
(g)	G(5,4)	G'(5,4)

5. For each of the following points,

A(3,5) B(-4,2) C(-5,-3) D(4,-5)

find the image for a translation of

(a) 3 units to the right,

(b) 2 units upward,

(c) 4 units to the left,

(d) 3 units downward.

6. Plot the following points and draw $\triangle$ RST.

R(-4,-3) S(3,-1) T(-1,5)

(a) Draw $\triangle$ R'S'T' if $\triangle$ RST is translated 2 units downward.

(b) Write the function that describes the translation.

7. Write the function that describes the translation.

(a) 7 units to the right
(b) 3 units down
(c) 4 units to the left
(d) 5 units up
(e) 1 unit to the right and 2 units down
(f) 8 units to the left and 1 unit up
(g) 2 units to the left and 4 units down

8. Let t be the translation 2 units to the right.

$$t : (x,y) \rightarrow (x + 2, y)$$

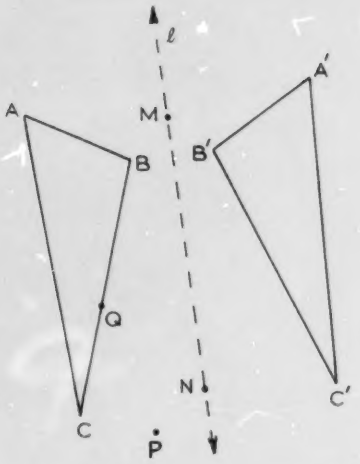
Let u be the translation 3 units upward.

$$u : (x,y) \rightarrow (x, y + 3)$$

Let v be the translation 2 units to the right and 3 units up.

(a) Find the image A' of A(4,-2) for the translation t.
(b) Find the image A'' of A' for the translation u.
(c) Now find the image of A(4,-2) for the translation v.
(d) What conclusion can you draw?

19-5. Reflections



The image $\triangle A'B'C'$ of $\triangle ABC$ was obtained as follows.

A copy of $\triangle ABC$ and line ℓ was made on a tracing sheet. The tracing sheet was flipped over and placed so that the copy of line ℓ coincided with the original line ℓ and with points M and N on ℓ fixed.

$\triangle A'B'C'$ was drawn from the new position of the copy of $\triangle ABC$.

The motion of the copy of $\triangle ABC$ has been identified earlier in Chapter 8 as a flip and line ℓ has been called the flip axis.

In the drawing above, think of the segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. Line ℓ is the perpendicular bisector of each of these segments. Why? Now use a tracing sheet and find the images P' and Q' of points P and Q for this motion. Is ℓ the perpendicular bisector of $\overline{PP'}$ and $\overline{QQ'}$?

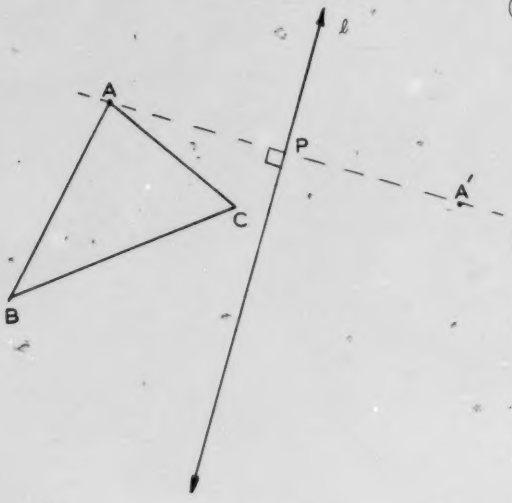
Imagine now a mirror placed upright perpendicular to the plane of the paper, along line ℓ facing $\triangle ABC$. If you look into the mirror from the left you will see $\triangle A'B'C'$ as a reflection of $\triangle ABC$. Point A' then is the reflection of point A. This is why we call the one-to-one correspondence of points and their images resulting from this motion, a reflection.

> Definition. A reflection in a given line ℓ is a rigid motion in which, for any point P not on ℓ and its image P', ℓ is the perpendicular bisector of $\overline{PP'}$. If P is on ℓ, then P is invariant ($P' = P$).

Exercises 19-5a

(Class Discussion)

1. Copy the drawing of △ ABC and line ℓ.



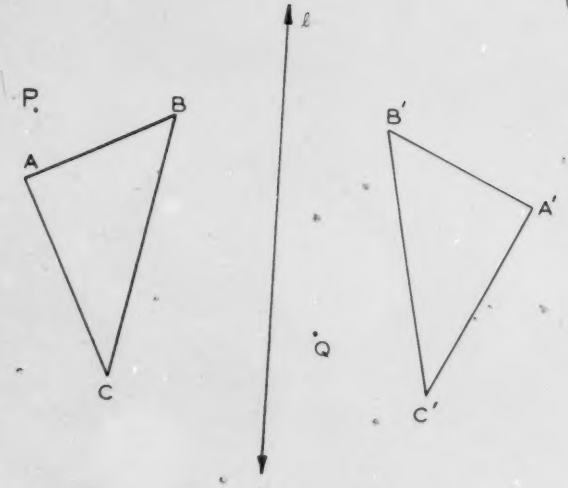
(a) Using only a ruler and compass, construct a line through A perpendicular to ℓ and intersecting ℓ at P. Locate A' on $\overleftrightarrow{AP}$ such that AP = A'P.

(b) Do the same to locate C' and B' and draw △ A'B'C'.

(c) Use a tracing sheet to show that △ A'B'C' is the flip image of △ ABC.

(d) Explain why △ A'B'C' is the image of △ ABC for the reflection in ℓ. (Use the definition of a reflection)

2. △ A'B'C' is the image of △ ABC for the reflection in ℓ. Copy this drawing.



(a) Find the images P' and Q' of points P and Q. Now find the images of P' and Q'. What is the image of △ A'B'C'? In general, for a reflection in ℓ, what is the image of the image of a point? For a reflection in ℓ, will a point and its image always be on opposite sides of the line ℓ? Discuss.

(b) Fold your drawing along line ℓ and describe the position of $\triangle ABC$ and $\triangle A'B'C'$. Do the points P and Q coincide with their images P' and Q'?

(c) For $\triangle ABC$, the trip from A to B to C is in a clockwise direction. In other words, the sense or orientation of $\triangle ABC$ is clockwise. What is the sense or orientation of $\triangle A'B'C'$? This suggests that for a reflection in ℓ, the orientation in a plane of a figure not on ℓ is the opposite of the orientation of its image.

3. Copy the closed figure and line ℓ as shown. Select two points A and B that lie on the figure and find the images of these points for the reflection in ℓ.

(a) Do A' and B' lie on the figure?

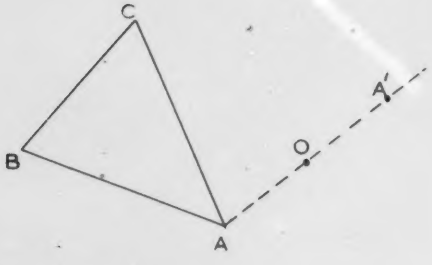
(b) For any point P on this figure, where will P' lie?

(c) Fold your drawing along line ℓ and describe the result.

(d) What is the image of this figure for the reflection in ℓ?

Since this figure is invariant for the reflection in ℓ, line ℓ is called an <u>axis of symmetry</u> for the figure. A figure is said to be <u>symmetric</u> if it has at least one axis of symmetry.

4. Copy the triangle, $\triangle ABC$, and point O shown below.



(a) Draw the line through points A and O, and locate A' on the line so that $A'O = AO$ as shown.

Point A' is said to be the image of point A for the <u>reflection in point O</u>.

(b) Find the image B' of point B by drawing the line $\overleftrightarrow{BO}$ and locate B' on $\overleftrightarrow{BO}$ so

that O is the midpoint of the segment $\overline{BB'}$.

(c) Find C' in the same way A' and B' were found.

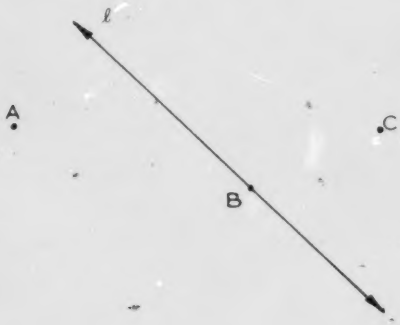
(d) Draw $\triangle A'B'C'$, the image of $\triangle ABC$ for the reflection in point O.

Definition. A reflection in a given point O is a rigid motion in which for any point P and its image P', O is the midpoint of $\overline{PP'}$. Point O is invariant for this reflection.

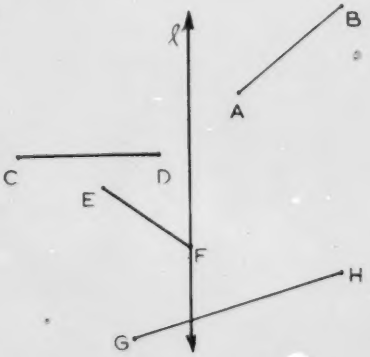
Exercises 19-5b

(Note: When necessary, copy the drawings and do the work on your drawings.)

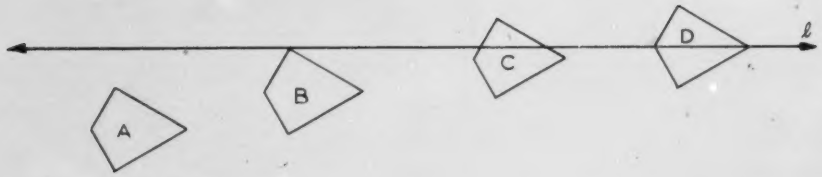
1. Find the images of points A, B, and C for the reflection in ℓ.



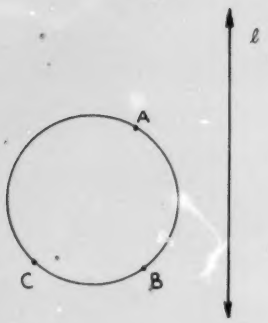
2. Find the images of the segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$ for the reflection in ℓ.



3. Find the images of figures A, B, C, and D for the reflection in ℓ.



4.



(a) Reflect the circle in the line ℓ and mark points A', B', C' on the image of the circle.

(b) On the circle, the trip from A to B to C is clockwise. On the image circle, what direction is the trip from A' to B' to C'?

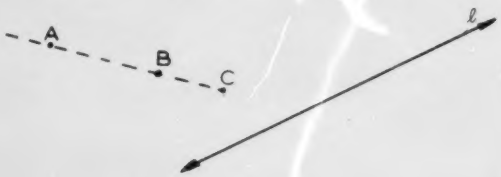
5.

(a) Draw a line m different from ℓ which is invariant for the reflection in ℓ.

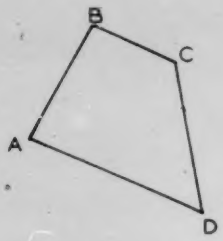
(b) Describe all the lines which are invariant for the reflection in ℓ.

6. A line m is parallel to a reflecting line ℓ. Describe the image m' of line m for the reflection in ℓ.

7. Points A, B, C are collinear. Show that their images A', B', and C' are collinear for the reflection in ℓ.

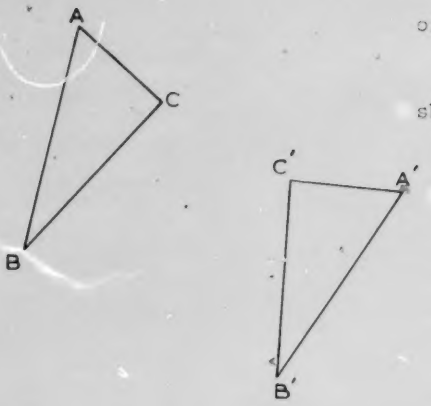


8. Find the image of quadrilateral ABCD for the reflection in point O.



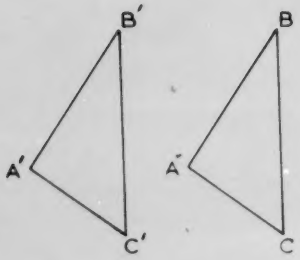
9. $\triangle A'B'C'$ is a reflected image of $\triangle ABC$.

Find the line of reflection, and show how you found it.



10. $\triangle A'B'C'$ is the image of $\triangle ABC$ for a rigid motion.

Explain why the rigid motion cannot be a reflection.



11. Find the lines of symmetry for each of the following figures.

(a)

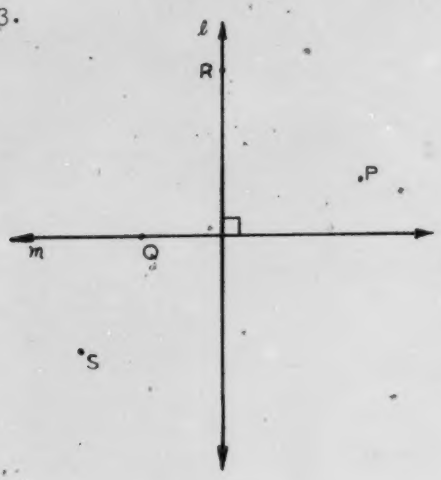
(b)

(c)

(d)

12. How many lines of symmetry does a circle have?

13.



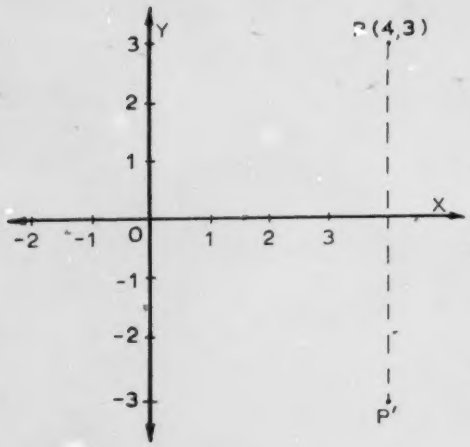
Lines ℓ and m are perpendicular, and points P, Q, R, S are located as shown.

Find the images of P, Q, R and S

(a) for the reflection in ℓ,

(b) for the reflection in m.

19-6. Reflections in the Coordinate Axes



Point P is located at (4,3).

To find its image P' for the reflection in the X-axis, the dotted line is drawn perpendicular to the X-axis.

Since P is 3 units above the X-axis, why must P' be 3 units below the X-axis?

Why must P' be 4 units to the right of the Y-axis?

The reflection of P about the X-axis is expressed as the following correspondence:

$$P(4,3) \rightarrow P'(4,-3).$$

Exercises 19-6a

(Class Discussion)

1. Plot points A, B, and C and their given images to verify that they are reflections in the X-axis.

   $A(-3,-2) \rightarrow A'(-3,2)$
   $B(5,-4) \rightarrow B'(5,4)$
   $C(-4,6) \rightarrow C'(-4,-6)$

   (a) What do you notice about the x-coordinates of the points and their images?

   (b) How are the y-coordinates of the images obtained?

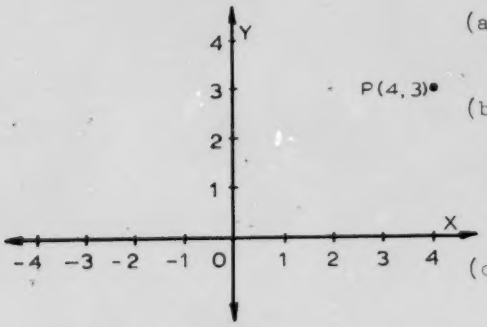
   In general, the reflection u of any point (x,y) in the X-axis is expressed as a function as follows:

   $$u : (x,y) \rightarrow (x,-y).$$

   (c) Use this definition to explain how the images of points A, B, and C above can be obtained.

   (d) Show how the definition above can be used to find the reflected image of point $(-6,-8)$ in the X-axis.

2. Copy the diagram below. To find the reflected image P' of P in the Y-axis, follow these steps.



   (a) Draw a dotted line through P perpendicular to the Y-axis.

   (b) How many units must P' be to the left of the Y-axis? Locate P' on the dotted line and write the coordinates of P'.

   (c) Express the reflection of P in the Y-axis as a correspondence.

3. Plot points A, B, and C and their images to verify that they are reflections in the Y-axis.

   $A(-3,-5) \rightarrow A'(3,-5)$
   $B(-4,6) \rightarrow B'(4,6)$
   $C(5,-4) \rightarrow C'(-5,-4)$

   (a) Explain how the coordinates of the images may be obtained from the coordinates of points A, B, and C.

   In general, the reflection v of any point (x,y) in the Y-axis

is expressed as a function as follows:

$$v : (x,y) \rightarrow (-x,y).$$

(b) Show how this definition can be used to find the reflected image of point $(-6,-8)$ in the Y-axis.

Exercises 19-6b

1. Find the coordinates of the images of these points for the reflection in the X-axis.

(a) $(2,-5)$	(d) $(-5,6)$	(g) $(-7,-4)$
(b) $(-3,-4)$	(e) $(0,6)$	(h) $(3,0)$
(c) $(3,-2)$	(f) $(-5,0)$	(i) $(0,-4)$

2. Find the coordinates of the images of the points listed in Exercise 1 for the reflection in the Y-axis.

3. Which of the following correspondences can be the result of the reflection in the X-axis? in the Y-axis? in neither axis?

(a) $(3,6) \rightarrow (3,-6)$	(e) $(4,0) \rightarrow (-4,0)$
(b) $(-2,5) \rightarrow (2,5)$	(f) $(7,3) \rightarrow (7,-2)$
(c) $(-4,-6) \rightarrow (-4,6)$	(g) $(-4,5) \rightarrow (4,-5)$
(d) $(0,5) \rightarrow (0,5)$	(h) $(2,3) \rightarrow (-2,3)$

4. Write the coordinates of three points that are invariant for the reflection in

(a) the X-axis (b) the Y-axis.

5. What point is invariant for the reflection both in the X-axis, and in the Y-axis?

6. Point A is located at $(-3,2)$ and point B is located at $(2,-3)$. Plot these points and draw the segment $\overline{AB}$.
Draw the image of $\overline{AB}$ for the reflection in

(a) the X-axis (b) the Y-axis.

7. The vertices of $\triangle ABC$ are located at $A(1,2)$, $B(3,6)$ and $C(5,2)$.

(a) Plot the triangle, $\triangle ABC$.

(b) Find the coordinates of the vertices of $\triangle A'B'C'$ for the reflection of $\triangle ABC$ in the X-axis, and plot $\triangle A'B'C'$.

(c) Find the coordinates of the vertices A", B", and C" of $\triangle A"B"C"$ for the reflection of $\triangle A'B'C'$ in the Y-axis, and plot $\triangle A"B"C"$.

(d) Compare the coordinates of A with A", B with B", C with C". What pattern do you see?

(e) Compare the orientation of $\triangle ABC$ with $\triangle A'B'C'$, and the orientation of $\triangle ABC$ with $\triangle A"B"C"$.

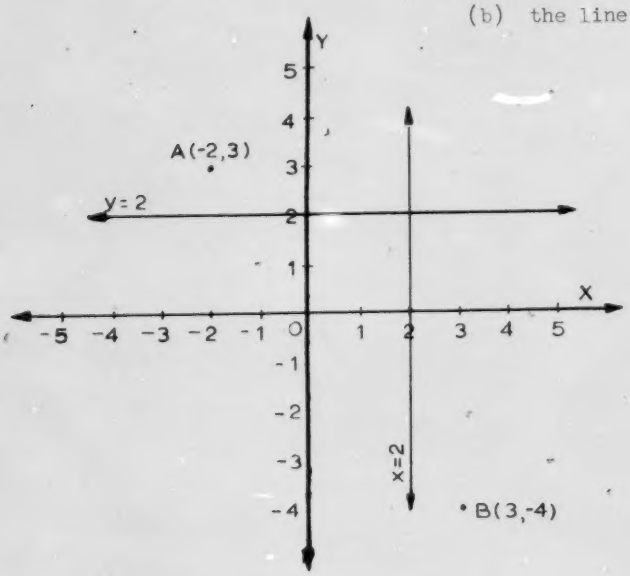
8. Write the coordinates of the end points A and B of a segment $\overline{AB}$ that is invariant for the reflection in

(a) the X-axis (b) the Y-axis.

9. Find the images of A(-2,3) and B(3,-4) for the reflection in

(a) the line $y = 2$,

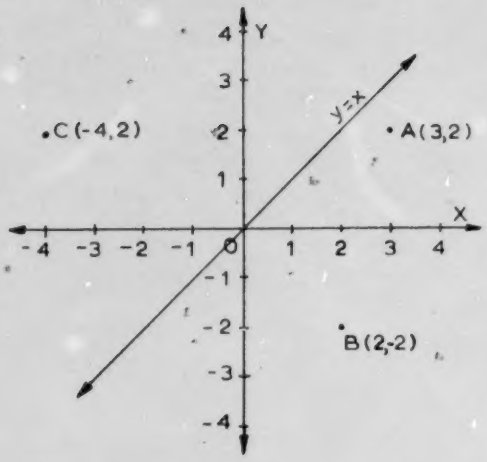
(b) the line $x = 2$.



10. Find the equations of the images of the two lines $x = 2$ and $y = 2$ for the reflection in the

(a) X-axis (b) Y-axis.

11.



Copy this diagram and find the images of $A(3,2)$, $B(2,-2)$ and $C(-4,2)$ for the reflection in the line $y = x$.

What would be the image of any point $P(a,b)$ for the reflection in the line $y = x$?

19-7. Rotations

Earlier in Chapter 8 you were introduced to rigid motions which were called turns or rotations. In this section we shall explore further these rigid motions.

Suppose that we have the ray $\overrightarrow{OP}$ given as shown.

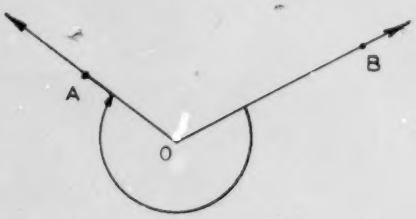
The image ray $\overrightarrow{OP'}$ of $\overrightarrow{OP}$ was obtained as follows.

A copy of the ray $\overrightarrow{OP}$ was made on a tracing sheet. The tracing sheet was then held down by a pencil point at O and turned counterclockwise as shown by the curved arrow so that the copy of $\overrightarrow{OP}$ marked the location where $\overrightarrow{OP'}$ was drawn.

The physical movement of the tracing sheet described above is called a turn, and the fixed point O for this turn is called the center of turn.

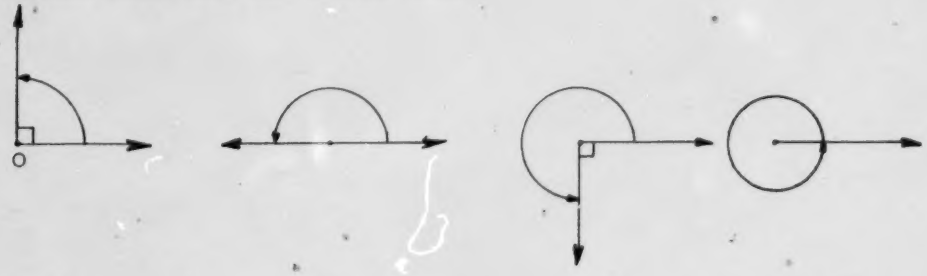
The curved arrow is drawn so that it indicates the direction of the turn, clockwise or counterclockwise, and so that it starts at the original ray and ends at the image ray. Therefore from the curved arrow we can tell which one of the two rays is the given ray and which one is the image of the given ray.

In the drawing below, the curved arrow indicates that the turn is clockwise about the fixed point O, and $\overrightarrow{OA}$ is the given ray with $\overrightarrow{OB}$ its image.



In this drawing, the curved arrow again indicates a clockwise turn about O, but this time $\overrightarrow{OB}$ is the given ray and $\overrightarrow{OA}$ is its image.

The diagrams below show some special turns.



I II III IV

Figure I shows a quarter turn.
Figure II shows a half turn.
Figure III shows a three-quarter turn.
Figure IV shows a full turn.

Unless otherwise indicated, the direction of these special turns is counterclockwise.

In Figures I and III, the rays are perpendicular. In Figure II, the rays are collinear. In Figure IV, the given ray is its own image.

Now to summarize, a turn is a motion of the tracing sheet which can be specified by a curved arrow, and by a given ray and its image with their common vertex at the center of turn.

We shall refer to the one-to-one correspondence of points and their images resulting from a turn as a rotation. The fixed point O then becomes the center of rotation.

In the following class discussion exercises we shall examine how points and their images are related in rotations.

Exercises 19-7a

(Class Discussion)

1. A counterclockwise rotation is defined by the curved arrow and the pair of rays at O.

Which ray is the given ray? the image ray?

Copy the rays, curved arrow, and point P.

To find the image P' of point P for this rotation about O, follow these steps.

(a) On a tracing sheet make a copy of point P and of the given ray with vertex O.

(b) Turn the tracing sheet keeping point O fixed so that the copy of the given ray coincides with the image ray.

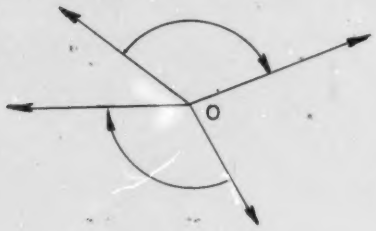
(c) The copy of P on the tracing sheet marks the location of the image P'. Mark the location of P'.

Now take a compass, and with center at O and radius OP, draw an arc to see if it passes through P'.

This shows that under a rotation, a point and its image can be connected by an arc of a circle with its center at the center of rotation.

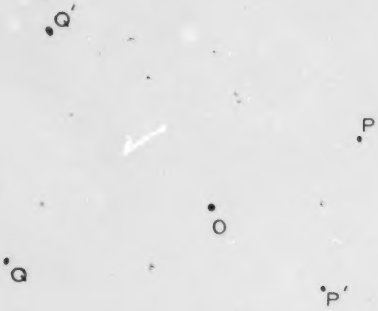
How does it also show that the distance of a point and of its image from the center of rotation is the same?

2. Two pairs of rays with their curved arrows are shown at O.



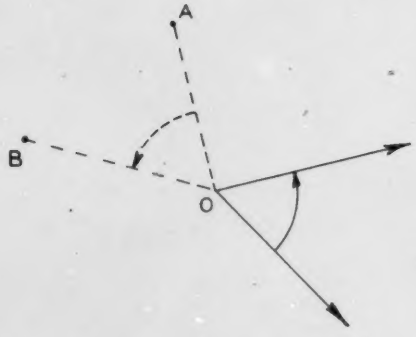
Describe a way of using a tracing sheet to determine if the two pairs of rays with their curved arrows specify the same turn about O.

3. Point P' is the image of P, and Q' is the image of Q, and point O is the center of rotation.



Describe a way of using a tracing sheet to determine if points P and Q are related to their images by the same turn about O.

4. A turn is specified by the curved arrow and two rays shown at O.



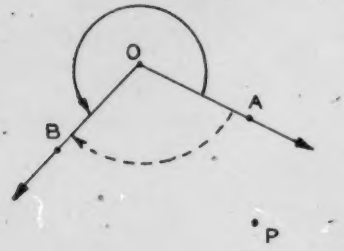
Use a tracing sheet to show that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with their curved arrow specify the same turn but B is not the image of A for this rotation about O.

This shows that two points may be related by a turn about a fixed point O but one is not the image of the other for this rotation unless their distances to the center of rotation are the same.

5.

Show that a rotation of a half turn about O and a reflection in O result in the same image of A.

6. Copy rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and point P as shown.

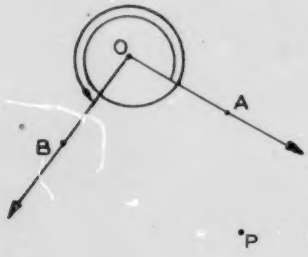


(a) Find P' the image of P by turning the tracing sheet counter-clockwise about O as shown by the solid curved arrow.

(b) Now find the image P" of P by turning the tracing sheet clockwise about O as shown by the dotted curved arrow.

What did you get as a result of the two turns? This shows that two different turns might result in the same rotation.

As a matter of fact, if we are permitted to continue beyond a full turn, there are many turns which result in the same rotation.



For example, the turn shown at the left results in the same rotation as the turns described above.

See if you can draw some other examples.

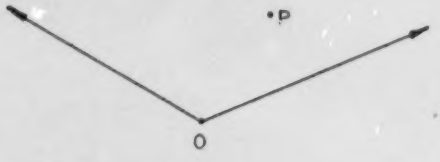
For our discussion, we shall limit ourselves to full turns or less.

In general, a rotation is a rigid motion in which all points and their images can be related by the same turn about a fixed point called the center of rotation, and any point and its image point are equidistant from the center of rotation.

Exercises 19-7b

(Note: When necessary, copy the drawings and do the work on your drawings.)

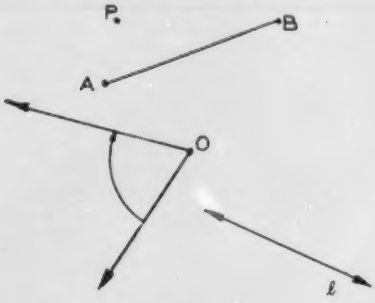
1. Make four copies of these two rays and use curved arrows to show four different turns (less than full turns) that can be determined using these two rays.



For each turn, find P', the image of P.

Identify the turns that result in the same rotation.

2. For the rotation shown at O, find



(a) P' the image of P

(b) $\overline{A'B'}$ the image of $\overline{AB}$

(c) ℓ' the image of ℓ.

Draw the arcs of circles which connect P with P', A with A', and B with B'.

3. Find the image of A for a

(a) quarter turn about O.

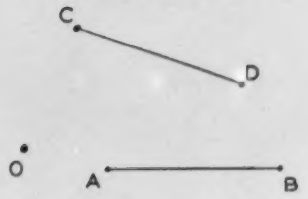
(b) half turn about O.

(c) three-quarter turn about O.

(d) full turn about O.

(e) reflection in O.

For what turn about O is A' the same as for a reflection in O?

4. 

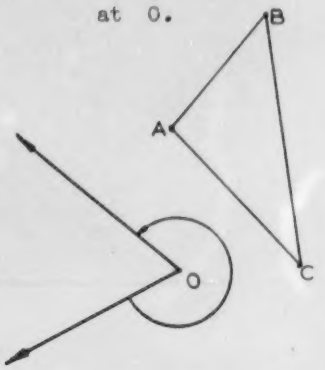
Find the image of $\overline{AB}$ and of $\overline{CD}$ for

(a) a quarter turn about O.

(b) a half turn about O.

(c) a reflection in O.

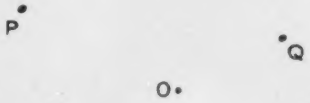
5. (a) Find $\triangle A'B'C'$, the image of $\triangle ABC$, for the rotation shown at O.



(b) In $\triangle ABC$, the trip from A to B to C is clockwise. In $\triangle A'B'C'$ what is the direction of the trip from A' to B' to C'?

(c) Using the same two rays at O, show with a curved arrow another turn which results in the same image $\triangle A'B'C'$.

6. Find the image of P and of Q for a full turn about O. Explain why a full turn results in the identity motion.



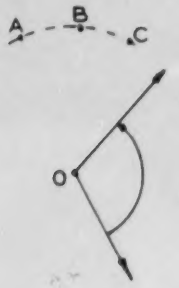
7. What point is always fixed for any rotation? Suppose a rotation has two fixed points. Explain why the rotation must be the identity motion.

8. Explain why Q cannot be an image of P for a rotation about O.

Find two different points that can be centers of rotations in which Q is the image of P.

Where are all the possible centers of rotations in which Q is the image of P?

9. Points A, B, C lie on an arc of a circle with center at O.



Find A', B', C' for the rotation shown at O.

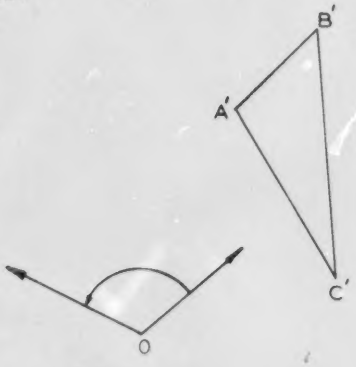
Describe where the image points lie.

10. Mark a point O and draw a circle which is invariant for all rotations about O. Describe all circles which are invariant for all the rotations about O.

11. Given the square shown below with point O at the intersection of the diagonals.

Describe three different turns about O for which the square is invariant.

12.



Given $\triangle A'B'C'$ and the rotation shown at O.

Find $\triangle ABC$ which has $\triangle A'B'C'$ as its image for this rotation.

BRAINBUSTER

$\overline{A'B'}$ is the image of $\overline{AB}$ for some rotation.

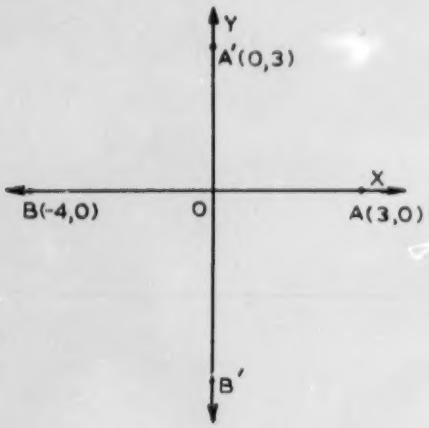
(a) Use a compass and straightedge to find the center of rotation.

(b) Draw a curved arrow and two rays at the center to specify the rotation.

19-8. Rotations and Coordinates

In this section we shall examine how rotations may be described in a coordinate system.

For convenience and simplicity we shall always take the origin as the center of rotation. Also at this time we shall consider only the special turns, i.e., quarter turns, half turns, and three-quarter turns, taken in a counterclockwise direction.



Point A(3,0) is given on the X-axis as shown.

For a quarter turn about O, the image A' must lie on the Y-axis as shown and OA' must equal OA. Why?

Then A' is located at (0,3).

Now consider B(-4,0) on the X-axis as shown.

For a quarter turn about O, where is B' located?

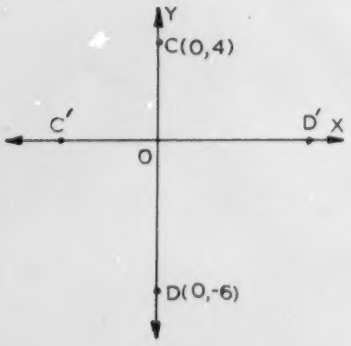
When we look at the coordinates of points A and B together with those of their images, we see that there is a pattern.

$A(3,0) \rightarrow A'(0,3)$

$B(-4,0) \rightarrow B'(0,-4)$

For a quarter turn about O, the image for any point (x,0) on the X-axis is the point (0,x) on the Y-axis. If we call this rotation f, we can write the following function:

$$f : (x,0) \rightarrow (0,x) .$$



Now let's consider points C(0,4) and D(0,-6) on the Y-axis as shown.

For a quarter turn about O, the images C' and D' will lie on the X-axis as shown such that $OC = OC'$ and $OD = OD'$.

$C(0,4) \rightarrow C'(-4,0)$
$D(0,-6) \rightarrow D'(6,0)$

What are the coordinates for C' and D'? When we compare the coordinates of C and D with those of their images, we again can see a pattern.

Now try some other points on the X-axis and find their images under this same rotation. For a quarter turn about O, the image for any point (0,y) on the Y-axis is (-y,0) on the X-axis. Since this is the same rotation f as before, we can write the following:

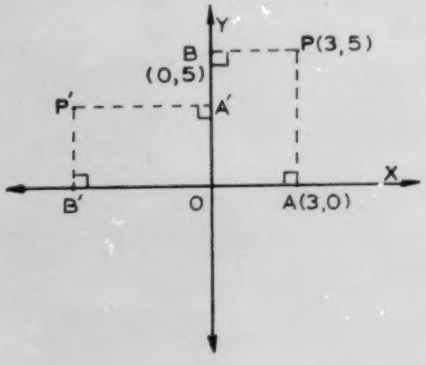
$$f : (0,y) \rightarrow (-y,0).$$

In the following class discussion exercises, we shall develop the general rule for finding the image of any point (x,y) for the rotation f resulting from a quarter turn about O.

Exercises 19-8a

(Class Discussion)

1. Point P(3,5) is given as shown. $\overline{PA}$ and $\overline{PB}$ are perpendicular respectively to the X- and Y-axes forming the rectangle PAOB.



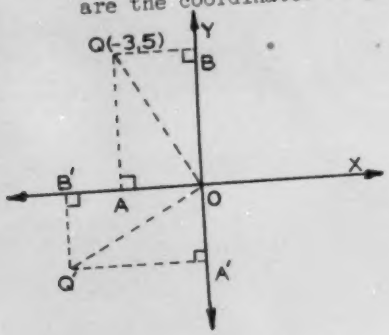
(a) Explain why the coordinates for A are (3,0), and for B are (0,5).

(b) Check with a tracing sheet to see that the image of the rectangle is as shown above for a quarter turn about O.

Explain why the coordinates for A' are (0,3), and for B' are (-5,0).

(c) Then why must the coordinates for P' be (-5,3)? Why must PO equal P'O?

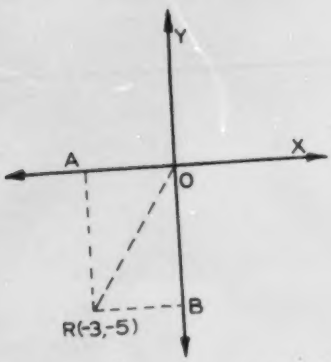
2. (a) Point Q(-3,5) and rectangle QAOB are given as shown. What are the coordinates of points A and B?



(b) The image of Q'A'OB' of the rectangle QAOB is also shown.

(c) What are the coordinates of A' and B'? Then what are the coordinates of Q', the image of Q?

3. (a) Copy this diagram showing point R(-3,-5) and rectangle RAOB. What are the coordinates of points A and B?



(b) Locate A' and B' for a quarter turn about O and draw the image rectangle.

(c) What are the coordinates of points A' and B'? Then what are the coordinates of R'?

4. Below are shown points P, Q, and R with their images P', Q', and R' for the rotation f resulting from a quarter turn about O.

$P(3,5) \rightarrow P'(-5,3)$
$Q(-3,5) \rightarrow Q'(-5,-3)$
$R(-3,-5) \rightarrow R'(5,-3)$

Express in words how the coordinates of P' are obtained from the coordinates of P.

Do the same for points Q' and R'. What is the image of point S(3,-5) for the rotation f?

5. You recall that the images of points on the X-axis and Y-axis for the rotation f can be found from the following expressions:

$$f : (x,0) \rightarrow (0,x),$$

$$f : (0,y) \rightarrow (-y,0).$$

If we combine the results of these two expressions we get the following

general expression for any point (x,y):

$$f : (x,y) \rightarrow (-y,x).$$

Check to see if this general rule works for finding the images of points P, Q, R, and S in Exercise 4.

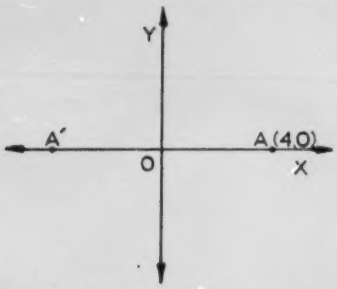
Exercises 19-8b

1. Find the images of the following points for a rotation f resulting from a quarter turn about O.

A(3,4) B(-2,3) C(5,2) D(-1,-4)

E(7,-5) F(-4,5) G(-4,-5) H(6,-4)

2. Point A is located at (4,0) as shown on the X-axis.



(a) For a half turn about O, explain why the image of A is $A'(-4,0)$.

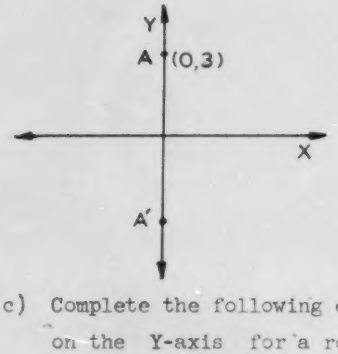
(b) Find the images of the following points on the X-axis for a half turn about O.

B(6,0) C(-3,0) D(-5,0)

(c) Write the general rule for finding images of points on the X-axis for a rotation g resulting from a half turn about O.

$$g : (x,0) \rightarrow ?$$

3. Point A (0,3) is given on the Y-axis as shown.



(a) Find the coordinates of A', the image of A for a half turn about O.

(b) Find the images of the following points on the Y-axis for a half turn about O.

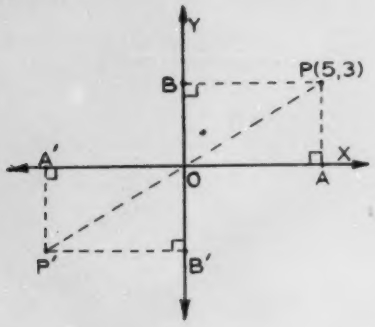
B(0,8) C(0,-5) D(0,-2)

(c) Complete the following expression for finding images of points on the Y-axis for a rotation g resulting from a half turn

about O.

$$g : (0,y) \rightarrow ?$$

4. Point P(5,3) and rectangle PAOB are given as shown.



(a) Find the coordinates of A', B', and P' of the image rectangle for a half turn.

(b) Draw coordinate axes and locate Q(-2,6). Draw the rectangle for Q and its image rectangle for a half turn. What are the coordinates for Q', the image of Q for a half turn?

(c) Describe in words how the coordinates of P' and Q' can be obtained from the coordinates of P and Q for a half turn.

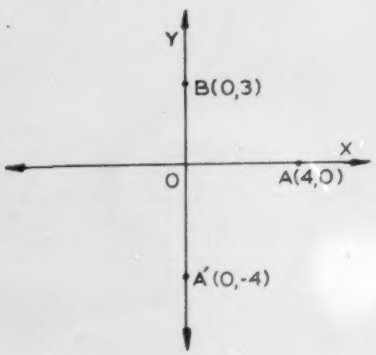
(d) The general rule for finding the image of any point (x,y) for a rotation g resulting from a half turn about O is as follows:

$$g : (x,y) \rightarrow (-x,-y).$$

Check to see if this rule works for points P and Q above. Show how the expressions you found in Exercises 2 and 3 can be combined to give the general rule above.

5. Find the images of the points given in Exercise 1 for a rotation g resulting from a half turn about O.

6. (a) Show how the coordinates of A', the image of A(4,0) are obtained for a three-quarter turn.



(b) Write the general rule for finding the images of points of the X-axis for the rotation h resulting from such a three-quarter turn.

$$h : (x,0) \rightarrow ?$$

(c) Find the coordinates of B', the image of B(0,3) for a three-quarter turn.

(d) Write the general expression for finding the images of points on the Y-axis for the rotation h resulting from a three-quarter turn.

$$h : (0,y) \rightarrow ?$$

(e) Show how the two expressions you obtained above can be combined to obtain the following expression for finding the image of any point (x,y) for the rotation h resulting from a three-quarter turn.

$$h : (x,y) \rightarrow (y,-x)$$

7. Find the images of the points given in Exercise 1 for the rotation h resulting from a three-quarter turn about O.

8. Develop the general rule for finding the image of any point (x,y) for a reflection in O. This rule is the same as the rule for what turn about O?

9. For each of the following, determine if the image is the result of a quarter turn, half turn, or three-quarter turn.

(a) $A(5,1) \rightarrow A'(-1,5)$

(b) $B(6,8) \rightarrow B'(-6,-8)$

(c) $C(6,3) \rightarrow C'(-3,6)$

(d) $D(7,2) \rightarrow D'(-7,-2)$

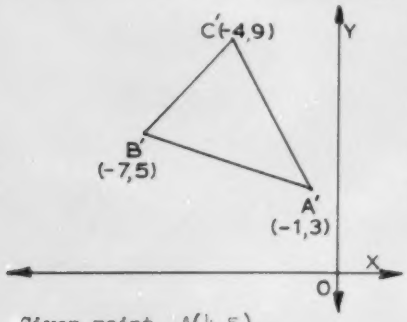
(e) $E(-4,7) \rightarrow E'(7,4)$

(f) $F(-6,3) \rightarrow F'(-3,-6)$

(g) $G(-3,-5) \rightarrow G'(-5,3)$

(h) $H(2,-4) \rightarrow H'(-2,4)$

10. Given $\triangle A'B'C'$ with $A'(-1,3)$, $B'(-7,5)$, and $C'(-4,9)$ as shown. Find $\triangle ABC$ which has $\triangle A'B'C'$ as its image for a



(a) quarter turn,

(b) half turn,

(c) three-quarter turn.

11. Given point $A(4,5)$,

(a) Find the image A' of A for a quarter turn.

(b) Find the image A'' of A' for a quarter turn.

(c) Find the image A''' of A for a half turn.

(d) What result did you get? Discuss.

19-9. Summary

Section 19-1.

Any rigid motion in a plane is determined by the three vertices of a triangle and the three corresponding vertices of the congruent image of a triangle.

Images for given figures under a rigid motion can be found using a tracing sheet. The sheet moves copies of original figures from their given location to the location of their images as prescribed by three non-collinear points and their corresponding images. Sometimes it is necessary to flip over the tracing sheet.

For a rigid motion, the original figure and its image are congruent.

Section 19-2.

A rigid motion is a function or a mapping of a plane onto itself which preserves distances.

Points or figures which are their own images are said to be invariant. It is possible that a figure may be invariant but the points of the figure are not.

Since for a rigid motion each point is the image of some point, a rigid motion results in a one-to-one correspondence of points in a plane.

A rigid motion in which each point is its own image is called an identity motion.

Section 19-3.

A translation is a rigid motion in which the distances between the points and their images are equal and the directions from the points to their images are the same.

Since a translation can be determined by a single point and its image, a translation can be designated by an arrow which indicates the distance and direction from each point to its image.

Section 19-4.

In a coordinate system a translation can be described compactly as a function.

$$t : (x,y) \rightarrow (x + 2, y + 3) \qquad u : (x,y) \rightarrow (x - 2, y - 3)$$

Translation t is a translation of 2 units to the right and 3 units upward. Translation u is a translation of 2 units to the left and 3 units downward.

Section 19-5.

A reflection in a given line ℓ is a rigid motion in which, for any point P not on ℓ and its image P', ℓ is the perpendicular bisector of $\overline{PP'}$. If P is on ℓ then P is invariant (P' = P),

A reflection in a line changes the orientation of the plane.

A reflection in a given point O is a rigid motion in which, for any point P and its image P', O is the midpoint of $\overline{PP'}$. Point O is invariant for the reflection.

If a figure is invariant for the reflection in a line ℓ, then ℓ is called the axis of symmetry for the figure.

A figure is said to be symmetric if it has at least one axis of symmetry.

For a reflection in a line, the image of the image of a point is the point itself.

Section 19-6.

The reflection u of any point (x,y) in the X-axis is expressed as follows: $u : (x,y) \rightarrow (x,-y)$.

The reflection v in the Y-axis for any point (x,y) is expressed as follows: $v : (x,y) \rightarrow (-x,y)$.

Section 19-7.

A turn is the motion of a tracing sheet specified by a given ray and its image ray with their common vertex at the center of turn, and by a curved arrow which starts at the given ray and ends at the image ray showing the direction of the turn.

A rotation is a rigid motion in which all points and their images can be related by the same turn about a fixed point called the center of rotation, and any point and its image point are equidistant from the center of rotation.

There are many different turns that result in the same rotation.

Some special rotations are those of a quarter turn, half turn, three-quarter turn, and full turn. A reflection in a point O and a rotation of a half turn about the same point O result in the same rigid motion. A rotation of a full turn is the identity motion.

Section 19-8.

In a coordinate system, the special rotations of a quarter turn, half turn, and three-quarter turn about the origin O are expressed as follows:

Quarter turn	$f : (x,y) \rightarrow (-y,x)$
Half turn	$g : (x,y) \rightarrow (-x,-y)$
Three-quarter turn	$h : (x,y) \rightarrow (y,-x)$

Slides, flips, and turns refer to the physical movement of the tracing sheet whereas translations, reflections, and rotations refer to the one-to-one correspondences of points.

Chapter 20

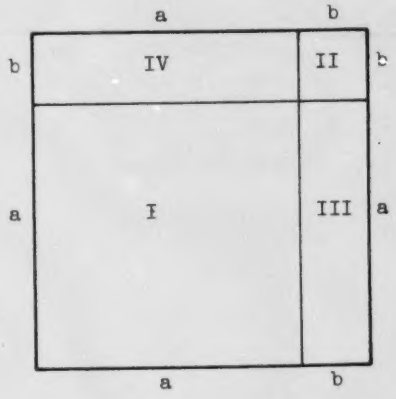
SQUARES AND RECTANGLES

20-1. Introduction

In this short chapter we shall become acquainted with several very important formulas, which will be used often in later chapters. These formulas are easily proved from the properties of addition and multiplication of real numbers. They also are easily pictured in connection with the areas of certain squares and rectangles. These pictures should help you to remember the formulas. We use the fundamental fact that the area of any rectangle is the product of the length of its base and its altitude. (In the case of the square, base and altitude are equal.)

20-2. A Formula for $(a + b)^2$

When you studied the Pythagorean property you drew a square region with side $a + b$ and consequently with area $(a + b)^2$. This region consists of two smaller square regions I and II and two congruent rectangular regions III and IV. The area of I is a^2 and the area of II is b^2. Each of III and IV has the area ab. Consequently



$$(1) \qquad (a + b)^2 = a^2 + 2\,ab + b^2.$$

We do not need to rely on the use of areas to obtain this result. We can use well-known properties of addition and multiplication of real numbers. In fact, by the distributive property

$$(a + b) \times (a + b) = [a \times (a + b)] + [b \times (a + b)]$$
$$= (a \times a) + (a \times b) + (b \times a) + (b \times b) \quad \text{(Why?)}.$$

What property do we now use to replace $b \times a$ by $a \times b$? Since

$$a \times a = a^2, \quad b \times b = b^2 \quad \text{and} \quad (a + b) \times (a + b) = (a + b)^2 \quad \text{(Why?)},$$

it is true that

$$(1) \qquad (a + b)^2 = a^2 + 2ab + b^2$$

for all real numbers a and b. In this proof, a or b can be negative or zero. Of course, in our geometric proof, negative values of a and b are excluded.

The result (1) is often useful in simplifying computations. Suppose for example that we wish to find the area of a square rug whose side is 15 ft. 1 in., that is $15\frac{1}{12}$ ft. We need to find $(15\frac{1}{12})^2$. The arithmetic is somewhat troublesome. If we think of $(15\frac{1}{12})^2$ as $(15 + \frac{1}{12})^2$ and apply our rule we obtain

$$225 + (2 \times 15 \times \tfrac{1}{12}) + \tfrac{1}{144} = 225 + 2.5 + \tfrac{1}{144}$$
$$\approx 227.5 \text{ sq. ft.}$$

The error is $\frac{1}{144}$ sq. ft. which is 1 sq. in. This error is negligible for most practical purposes.

To take another example, suppose that we wish to calculate $(4.12)^2$. We write $(4 + .12)^2 = 4^2 + (2 \times 4 \times .12) + (.12)^2$

$$= 16 + .96 + .0144 = 16.9744.$$

Exercises 20-2

1. Find $(a + 3)^2$ by using equation (1) and also geometrically by separating a square region, $a + 3$ on a side, into appropriate regions I, II, III and IV as in the text.

2. Approximate $(12.1)^2$. Find its exact value. What is the error committed in using the approximation?

3. In Exercise 2 draw a figure with regions I, II, III and IV as in the text. Do this on ruled paper. What are the areas of the four regions? What region are you neglecting in using the approximation?

4. Find quickly

$$101^2, \quad 61^2, \quad 10300^2, \quad (2.03)^2, \quad 4001^2.$$

5. Write 199 as $200 - 1 = 200 + (-1)$ and use equation (1) to find 199^2.

6. Draw a rectangle with base $a + 3$ and altitude $a + 2$. Show how the rectangular region can be divided into four parts to illustrate the result of multiplying $a + 3$ by $a + 2$.

7. Repeat Exercise 6 with dimensions $a + 4$ and $a + 1$.

20-3. A Formula for $(a - b)^2$ and a Related Result

From the result

(1) $$(a + b)^2 = a^2 + 2ab + b^2$$

found in the last section it is easy to get a result for $(a - b)^2$. Since $a - b$ may be written as $a + (-b)$,

$$(a - b)^2 = [a + (-b)]^2 = a^2 + 2a(-b) + (-b)^2.$$

Therefore

(2) $$(a - b)^2 = a^2 - 2ab + b^2.$$

For example, to square 98, we may write $98 = 100 - 2$ and conclude that

$$\begin{aligned} 98^2 &= 100^2 - (2 \cdot 2 \cdot 100) + 4 \\ &= 100^2 - 400 + 4 \\ &= 100\,(100 - 4) + 4 \\ &= (96 \times 100) + 4 \\ &= 9604. \end{aligned}$$

It is not very easy to picture

(2) $(a - b)^2 = a^2 - 2ab + b^2$.

However, let us mark a square of side $a + b$ as follows. Note that a and b alternate. Now construct a rectangle with sides a and b in the lower right-hand corner of Figure 2. If we draw three copies of this rectangle in the remaining corners we end with Figure 3.

Fig. 1

The shaded region in the center is a square region with side $a - b$ as you can easily show.

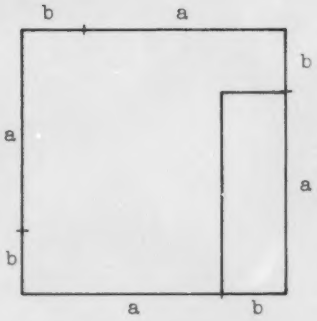


Fig. 2

The large square has the area $(a + b)^2$. The shaded square region has the area $(a - b)^2$ and each of the four rectangular regions which surround it has the area ab. Therefore

(3) $\quad (a - b)^2 = (a + b)^2 - 4ab.$

Now if we replace $(a + b)^2$ by $a^2 + 2ab + b^2$ and simplify we have

$$(a - b)^2 = a^2 - 2ab + b^2.$$

We may write (3) in the equivalent form:

$$(a + b)^2 = 4ab + (a - b)^2.$$

Now if we divide by 4, we obtain

(4) $$(\frac{a + b}{2})^2 = ab + (\frac{a - b}{2})^2.$$

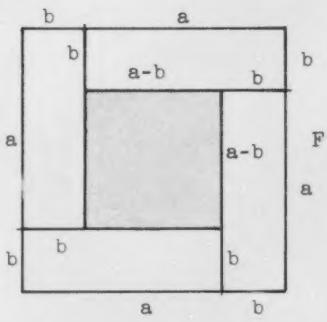


Fig. 3

It follows that

(5) $$(\frac{a + b}{2})^2 \geq ab$$

where the equality occurs only when $a = b$.

How can we interpret this result? Let us take any two positive numbers a and b. Their average is $\frac{a + b}{2}$ and their product is ab. Then (5) tells us that if a and b are different, the square of the average is greater than the product. Equation (4) tells us how much greater $(\frac{a + b}{2})^2$ is than ab.

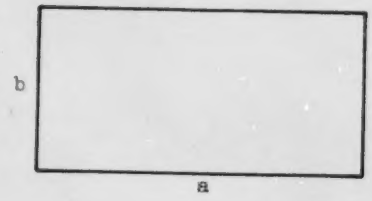
For example, let us take $a = 4$ and $b = 2$.

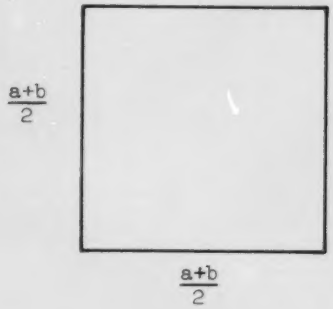
Since $\frac{a + b}{2} = 3$,

$$(\frac{a + b}{2})^2 = 9,$$

which is greater than $ab = 8$. The difference should be $(\frac{a - b}{2})^2 = (\frac{4 - 2}{2})^2$ and of course it is.

Let us interpret (5) geometrically. Suppose that we have a rectangle of base a and altitude b. The perimeter of this rectangle is $2a + 2b$. What is its area? Now consider a square with the same perimeter, $2a + 2b$. Each side must be $\frac{2a + 2b}{4} = \frac{a + b}{2}$. What is the area of this square?





According to (5), if a is different from b

$$(\frac{a + b}{2})^2 > ab.$$

This means that the area of the square is greater than the area of the rectangle. Of course, if $a = b$ we had a square to begin with and the square with side $\frac{a + b}{2}$ is identical with it.

Exercises 20-3

1. On a square of side 8, mark off segments $a = 5$ and $b = 3$ as shown in Figure 1. Draw the four rectangles and the shaded square region enclosed by them. Verify from your figure that

$$(5 + 3)^2 = [4 \times (5 \times 3)] + (5 - 3)^2.$$

2. Sketch a square of side $(a + 3)$ where $a > 3$, and mark off segments of length a and 3 alternately. Draw the four rectangles and the enclosed square. Write $(a + 3)^2$ so as to show how the large square region is built up of rectangular regions and a square region.

3. Find each of the following quickly:

$$49^2, \quad 998^2, \quad (.193)^2.$$

4. Find the area of a circle of radius 1.98. (Leave the result in terms of π, where $A = \pi r^2$.)

5. Approximate $(.99)^2$. Find its exact value. What error is committed in using the approximation?

6. Verify Equation (4) for

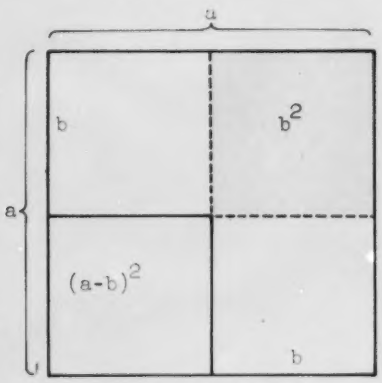
(a) $a = 3$, $b = 2$

(b) $a = 11$, $b = 9$

(c) $a = 1.1$, $b = .9$

(d) $a = \frac{10}{7}$, $b = \frac{7}{5}$

7. Show geometrically that $(a - b)^2 = a^2 - 2ab + b^2$ from the following figure,



by subtracting from area a^2 of the large square, the area of two rectangles with sides a and b. How do you compensate for the fact that you took away too much?

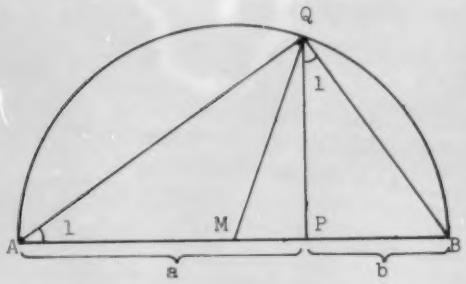
20-4. Another Picture

In the previous section, we proved that for positive real numbers a and b and $a > b$,

(1) $$(\frac{a + b}{2})^2 = ab + (\frac{a - b}{2})^2.$$

We also pictured this result in terms of a rectangle and a square with the same perimeter.

There is another geometric interpretation of (1). Let us place line segments of length a and b end-to-end. The midpoint M of the combined segment $\overline{AB}$ is at the distance $\frac{a+b}{2}$ from the endpoints A and B. Let us draw a semicircle with center at M and radius $\frac{a+b}{2}$. Next draw $\overline{PQ}$ perpendicular to $\overline{AB}$ at the point P where the segments meet. In the right triangle MPQ we know from the Pythagorean property that



$$(MQ)^2 = (MP)^2 + (PQ)^2.$$

Since $MQ = \frac{a+b}{2}$, the radius, and $MP = \frac{a+b}{2} - b = \frac{a-b}{2}$,

$$(\frac{a+b}{2})^2 = (\frac{a-b}{2})^2 + (PQ)^2.$$

But according to (1)

$$(\frac{a+b}{2})^2 = (\frac{a-b}{2})^2 + ab.$$

It must be true then that

$$(PQ)^2 = ab.$$

Let us verify this result in another way.

Exercises 20-4a

(Class Discussion)

1. What kind of triangle is AQB ? Why?

2. Show that the angles marked 1 are congruent and that $\triangle APQ \sim \triangle QPB$.

3. $\frac{AP}{PQ} = \frac{?}{PB}$

4. $(PQ)^2 = ? \cdot PB = a \cdot ?$

Exercises 20-4b

1. In the figure in the text why is $(MQ)^2 \geq (PQ)^2$? What does this inequality mean in terms of a and b? When can equality occur?

2. Draw a semicircle like that in the text using $a = 2$ and $b = 1$. Find the lengths of $\overline{PQ}$, $\overline{AQ}$, $\overline{BQ}$. Prove that triangles APQ and QPB are similar by showing that their sides are proportional. (Note that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.)

20-5. A Formula for $a^2 - b^2$

If we multiply the sum of two numbers $(a + b)$ by their difference $(a - b)$, we obtain

$$(a + b) \times (a - b) = a^2 - b^2. \tag{1}$$

This is a very handy result. For example

$$(100 + 2) \times (100 - 2) = 10{,}000 - 4$$

so that

$$102 \times 98 = 9996$$

with almost no work!

It is easy to see geometrically that (1) is true for all positive real numbers a and b with $a > b$. In Figure 1, $a^2 - b^2$ is the area of I + II (shown shaded).

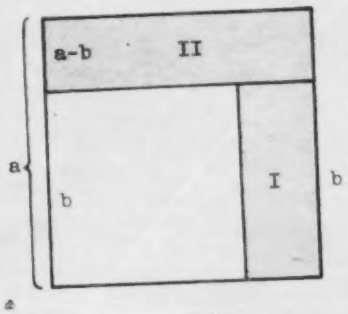


Figure 1

By moving I to the position shown in Figure 2 it is clear that I + II is equivalent to a rectangular region with base $a + b$ and altitude $a - b$ and therefore with area $(a + b) \times (a - b)$. Consequently

(1) $$a^2 - b^2 = (a + b) \times (a - b).$$

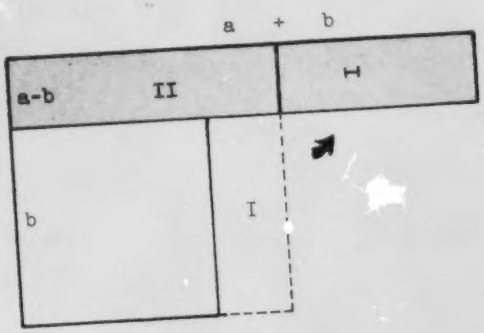


Figure 2

Exercises 20-5

1. Justify the result (1) by using the properties of addition and multiplication for real numbers.

2. Use (1) to find the following products quickly:

$$103 \times 97$$

$$9 \times 11$$

$$87 \times 93$$

$$10{,}001 \times 9999.$$

3. Simplify $(x + 2)^2 - (x - 2)^2$ where x is a real number greater than 2. Hint: Think of $x + 2$ as a single number a and $x - 2$ as a single number b and apply (1).

4. Simplify $(x + y)^2 - (x - y)^2$ by the method of Exercise 3.

5. In Section 20-3, we got the result

$$\left(\frac{a + b}{2}\right)^2 = ab + \left(\frac{a - b}{2}\right)^2.$$

Write this in the equivalent form

$$(\frac{a + b}{2})^2 - (\frac{a - b}{2})^2 = ab.$$

Show that this result may be proved from

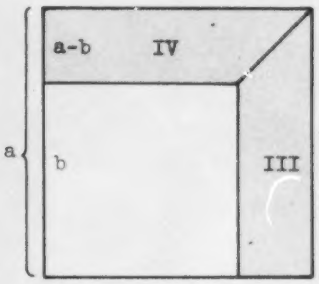
$$u^2 - v^2 = (u + v) \times (u - v),$$

which is (1) with the letters changed.

6. Show that

$$a^2 - b^2 = (a + b) \times (a - b)$$

by using the following figure:



and moving the trapezoidal region III in a suitable way.

20-6. Solving Some Problems

The results of this chapter can be used to solve certain problems.

Example. A rectangle with area 16 sq. ft. has a base which is 6 ft. longer than its height (Figure 1). What are its dimensions?

Method 1.

$$(x + 6)x = 16.$$

Apply the formula

$$(\frac{a + b}{2})^2 = ab + (\frac{a - b}{2})^2$$

with $a = x + 6$ and $b = x$.

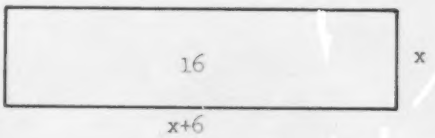


Figure 1

Then

$$ab = (x + 6)x = 16,$$

$$\frac{a + b}{2} = x + 3 \quad \text{and} \quad \frac{a - b}{2} = 3.$$

Hence

$$(x + 3)^2 = 16 + 9 = 25,$$

and

$$(x + 3) = 5 \quad \text{or} \quad -5.$$

Of course, $x + 3 = -5$ is impossible. Why? Hence

$$x + 3 = 5$$

and

$$x = 2.$$

Method 2. Since $(x + 6)x = x^2 + 6x$

$$x^2 + 6x = 16.$$

Draw a square of side x and area x^2 (Figure 2). Think of $6x$ as the area of two rectangular regions with sides x and 3. Add these regions to the square region as follows (Figure 3). The total area $x^2 + 6x$ is to be 16.

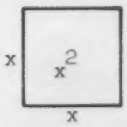


Figure 2

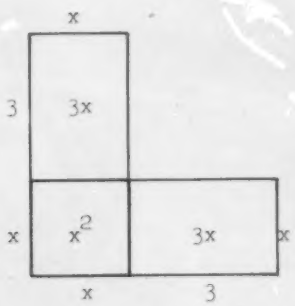


Figure 3

We can complete a square by adding the small shaded square region of area 9 (Figure 4). This gives

$$x^2 + 6x + 9 = (x + 3)^2$$

for the area of the large square. Since this area must be

$$25 + 5 = 25,$$

$$(x + 3)^2 = 25.$$

Then, as before,

$$x + 3 = 5$$

and

$$x = 2.$$

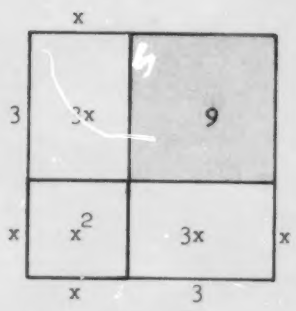
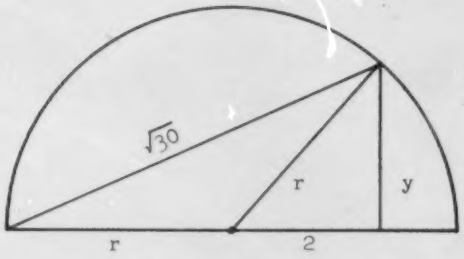


Figure 4

Exercises 20-6

1. A rectangle has area 15 sq. in. Its base is 2 in. longer than its height. Find its dimensions.

2. In the figure of a semi-circle shown, find the value of the radius r.



Hint: Use the Pythagorean property twice.

3. A certain rectangle has the perimeter 20 ft. Its base is 2 ft. longer than its height. What is its area?

20-7. Summary

In this chapter we have established four important relations. If a and b are any two real numbers, then

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(\frac{a + b}{2})^2 = ab + (\frac{a - b}{2})^2$
4. $a^2 - b^2 = (a - b)(a + b)$

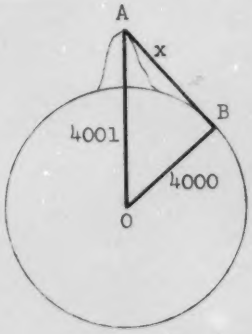
We have pictured these relations by using the areas of squares and rectangles. We have proved them by using tne familiar commutative, associative and distributive properties of the real numbers. These four relations will be used frequently in later chapters of this book so that the student should know them well.

Chapter 21

SQUARE ROOTS AND REAL NUMBERS

21-1. Introduction

You are standing on the top of a mountain one mile high, rising above a level plain. How far is it to the horizon?



We think of the earth as a sphere and the mountain as a bump on it. This of course is a simplified picture or mathematical model. The radius of the earth is about 3959 miles. Let us round this off to 4000 miles.

We draw a figure in which the distance x miles to the horizon is one leg of a right triangle. The other leg is 4000 miles and the hypotenuse is 4001 miles.

The Pythagorean property tells us that $(AO)^2 = (OB)^2 + (AB)^2$ so that

$$(4001)^2 = (4000)^2 + x^2$$

and so

$$x^2 = (4001)^2 - (4000)^2.$$

Of course we could square 4001 and 4000 and subtract. But, as you know, there is an easier way. Since $a^2 - b^2 = (a - b)(a + b)$, $(4001)^2 - (4000)^2 = (4001 - 4000) \times (4001 + 4000) = 1 \times 8001 = 8001$. Check by squaring to see that this is correct. If $x^2 = 8001$, what is x? It is $\sqrt{8001}$ of course. But how do we find this square root?

The problem of finding square roots is the subject of this chapter. Here we shall just make a start by finding a rough approximation to $\sqrt{8001}$. Is there a whole number answer? If 8001 were changed to 8100, we would have a "perfect square"; that is, we could write $8100 = 90 \times 90 = 90^2$. So $\sqrt{8100} = 90$. This is certainly too big. On the other hand, 80 is

much too small to be the square root of $\sqrt{8001}$. You can see this by multiplying 80 by 80. So $\sqrt{8001}$ is between 80 and 90. We write

$$80 < \sqrt{8001} < 90.$$

But $\sqrt{8001}$ is much nearer to 90 than to 80, wouldn't you say? Suppose that we try 88. Since $88^2 = 7744$, which is less than 8001, we know that

$$88 < \sqrt{8001} < 90.$$

To locate the square root more closely it is natural to try 89. Since $89^2 = 7921$, 89 is too small and we see that

$$89 < \sqrt{8001} < 90.$$

There is <u>no</u> whole number whose square is 8001 . But we have located the distance to the horizon within a mile; this is probably accurate enough for most purposes. However, as we shall see, it is possible to find as accurate values for square roots as we wish. Fortunately this can be done very simply without resorting to trial and error methods. We shall soon see how to do this.

<u>Exercises 21-1</u>

1. To find the value of 4001^2 in $4001^2 - 4000^2$ we <u>could</u> write $4001 = 4000 + 1$ and square this sum. Carry out the arithmetic and check with the value found in the text. Is it necessary to know what 4000^2 is equal to?

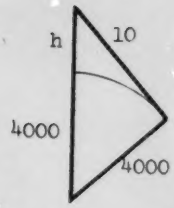
2. In the problem discussed in this section use 3959 miles instead of 4000 miles for the earth's radius. What square root needs to be found in this case? Estimate its value.

(In the following exercises use 4000 miles for the earth's radius.)

3. If the mountain in the problem were $\frac{1}{5}$ of a mile high, about how far would it be to the horizon? (Replace $8000\frac{1}{5}$ by 8000.)

4. How far could you see from a height of 250 feet? (Call this $\frac{1}{20}$ of a mile.)

5. How high above the ocean would you have to be to see 10 miles to the horizon?



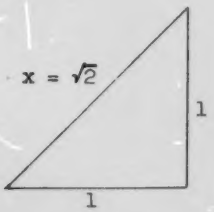
Hint: Use the figure. In squaring $(4000 + h)$ neglect h^2. Give the answer in miles and feet.

6. Suppose that your eyes are 5 feet above the ground. How far can you see? (Assume that 5 feet $= \frac{1}{1000}$ mile.)

7. How far can you see from a mountain 2 miles high?

21-2. Computation of Square Roots

It is useful to be able to compute square roots quickly. The method of doing so will be explained in connection with $\sqrt{2}$. This number occurs naturally as the hypotenuse of an isosceles right triangle with legs equal to 1. By the Pythagorean property,

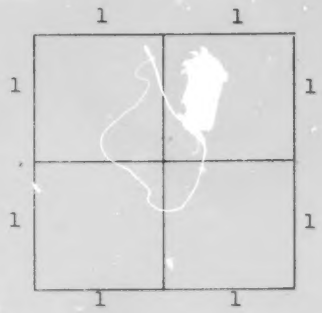


$$x^2 = 1^2 + 1^2 = 1 + 1 = 2.$$

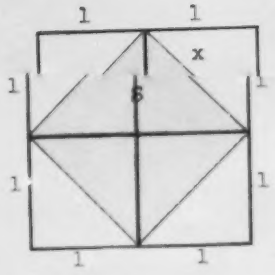
and hence

$$x = \sqrt{2}.$$

Even if we had never heard of the Pythagorean property we could easily see that in the figure, $x = \sqrt{2}$. Consider a square with side two units. We can draw four unit squares as shown so that the total area is certainly 4.

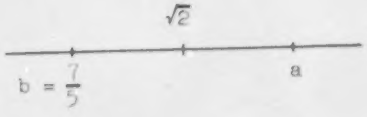


Now if we draw diagonals of the unit squares, we obtain the shaded square region S. Since each unit square has been cut in half, the area of S is 2. But the area of S must be x^2 if x is the length of one of its sides. Hence $x^2 = 2$ and $x = \sqrt{2}$.



How shall we compute $\sqrt{2}$? It is surely true that $1 < \sqrt{2} < 2$ since $1^2 < 2$ and $2 < 2^2$. If we try various rational numbers between 1 and 2 , it will soon be discovered that $\frac{7}{5}$ is a fair approximation to $\sqrt{2}$, since $(\frac{7}{5})^2 = \frac{49}{25}$, and $2 = \frac{50}{25}$. We see however that $\frac{7}{5}$ is too small.

It would be useful to bracket $\sqrt{2}$ between two rational numbers, one of which (like $\frac{7}{5}$) is too small and the other of which is too large. It would, of course, be still nicer if we could find a rational number which is <u>exactly</u> equal to $\sqrt{2}$. You remember perhaps that there is no such rational number, so that we must be satisfied with approximations. However, if you have forgotten that $\sqrt{2}$ is irrational, we shall soon prove that this is true.



Now there is an easy way to bracket $\sqrt{2}$ between $\frac{7}{5}$ and a number, a. We simply divide 2 by $\frac{7}{5}$, obtaining

$$a = \frac{2}{\frac{7}{5}} = 2 \times \frac{5}{7} = \frac{10}{7} .$$

You should verify that $\frac{10}{7}$ is, in fact, greater than $\sqrt{2}$ by showing that

$$(\frac{10}{7})^2 > 2.$$

So

$$\frac{7}{5} < \sqrt{2} < \frac{10}{7} .$$

We can see without squaring $\frac{10}{7}$ that it must be greater than $\sqrt{2}$. For we chose $a = \frac{5}{\frac{7}{5}}$.

Then

$$\frac{7}{5} \times a = 2.$$

Since

$$\sqrt{2} \times \sqrt{2} = 2,$$

if $\frac{7}{5} < \sqrt{2}$, a must be $> \sqrt{2}$.

The interval from $\frac{7}{5}$ to $\frac{10}{7}$ has a length of $\frac{1}{35}$ since $\frac{10}{7} - \frac{7}{5} = \frac{50 - 49}{35}$. We have therefore not located $\sqrt{2}$ on the number line very precisely.

There is a simple method of locating $\sqrt{2}$ within a shorter interval. We average the two coordinates and obtain:

$$\frac{\frac{7}{5} + \frac{10}{7}}{2} = \frac{99}{70}.$$

This is closer to $\sqrt{2}$ than either $\frac{7}{5}$ or $\frac{10}{7}$. In fact $(\frac{99}{70})^2 = \frac{9801}{4900}$ is very close to 2 since $2 = \frac{9800}{4900}$. Nevertheless it is a little too big. If we divide 2 by $\frac{99}{70}$ we obtain $\frac{140}{99}$ which is slightly too small.

The length of the interval is only $\frac{99}{70} - \frac{140}{99} = \frac{1}{70 \times 99} = \frac{1}{6930}$. By averaging again and dividing, we could narrow the interval still further. But we shall stop at this point and apply the process to some other examples.

Exercises 21-2

1. Show that $\frac{140}{99} < \sqrt{2}$.

2. Show that $\sqrt{3} < 1\frac{3}{4} = \frac{7}{4}$. Divide 3 by $\frac{7}{4}$ and show that the result is too small. Mark on the number line an interval within which $\sqrt{3}$ must lie. How long is this interval?

3. Average the numbers found in Exercise 2 and show that this average is slightly too large. Find 3 divided by $\frac{27}{56}$ and show that this result is too small. How long is the interval within which you have located $\sqrt{3}$?

4. Show that $\sqrt{5} < 2\frac{1}{4} = \frac{9}{4}$. Divide 5 by $\frac{9}{4}$ and show that the result is too small. Mark on the number line an interval within which $\sqrt{5}$ must lie. How long is this interval?

5. Average the numbers found in Exercise 4 and show that this average is slightly too large. Find 5 divided by $\frac{161}{72}$ and show that this result is too small. How long is the interval within which you have located $\sqrt{5}$.

6. In estimating the square root of 8001 in the first section, we found that $a = 90$ is too big. Use the divide and average method to obtain a better estimate.

7. Start with $\frac{3}{2}$ instead of $\frac{9}{5}$ as a first estimate for $\sqrt{2}$. Divide, average and divide to bracket $\sqrt{2}$ in an interval of length $\frac{1}{204}$.

21-3. Irrational Square Roots

We have been calculating such square roots as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$. Starting with a rational number we have found in each case that we could obtain excellent approximations, but in no case were we able to obtain a rational number which is an exact answer. Is this because we did not divide and average enough times? Or is it perhaps because we did not start with a good enough first approximation? The answer to both questions is No. There are no rational numbers equal to $\sqrt{2}$, $\sqrt{3}$, or $\sqrt{5}$.

Let us see how we can show this. Suppose, for example, that there is a rational number $\frac{p}{q} = \sqrt{2}$. Then if we divide 2 by $\frac{p}{q}$ the result $(\frac{2q}{p})$ must be equal to $\frac{p}{q}$. That is

(1) $$\frac{p}{q} = \frac{2q}{p} .$$

We can surely assume that $\frac{p}{q}$ is written in lowest terms, because if p and q had a common factor we could replace the fraction by an equal fraction with a smaller numerator and a smaller denominator. For example, if (1) were true with $\frac{p}{q} = \frac{14}{10}$, we could replace it by $\frac{7}{5}$ which has the same value.

Now if $\frac{p}{q}$ is in lowest terms, any fraction which is equal to $\frac{p}{q}$ must be obtained by multiplying p and q by the same integer, say k. For

example, the only positive fractions equal to $\frac{7}{5}$ are

$$\frac{7 \times 2}{5 \times 2} = \frac{14}{10}, \quad \frac{7 \times 3}{5 \times 3} = \frac{21}{15}, \quad \frac{7 \times 4}{5 \times 4} = \frac{28}{20},$$

or generally

$$\frac{7 \times k}{5 \times k}.$$

Consequently, if

$$\frac{p}{q} = \frac{2q}{p},$$

the <u>denominator</u> of $\frac{2q}{p}$ must be some integer times the denominator of $\frac{p}{q}$. That is, $p = (1 \times q)$ or $(2 \times q)$ or $(3 \times q)$, ... which means that $\frac{p}{q} = 1$ or 2 or 3 or ... So, if $\frac{p}{q}$ is a rational number for which $\frac{p}{q} = \sqrt{2}$, $\frac{p}{q}$ must be an integer. But $\sqrt{2}$ is <u>not</u> an integer! In fact,

$$1 < \sqrt{2} < 2.$$

We have proved that $\sqrt{2}$ is an <u>irrational</u> number, since it is not a rational number.

<u>Exercises 21-3</u>

1. Prove similarly that $\sqrt{3}$ is irrational.
2. Prove that $\sqrt{5}$ and $\sqrt{6}$ are irrational.
3. Why doesn't this method of proof show that $\sqrt{4}$ is irrational?
4. Show that $\sqrt{n}$ is irrational for any positive integer n which is not a perfect square.
5. Is it possible for $\frac{1}{2}\sqrt{2}$, $\frac{1}{3}\sqrt{2}$, $\frac{2}{3}\sqrt{2}$ to be rational?
   <u>Hint</u>: Assume that $\frac{1}{2}\sqrt{2} = \frac{r}{s}$. What does this say about $\sqrt{2}$?
6. Show that $\frac{p}{q}\sqrt{2}$ is irrational for all positive integers p and q.
7. Is it possible for $\frac{p}{q}\sqrt{3}$, $\frac{p}{q}\sqrt{5}$, or $\frac{p}{q}\sqrt{6}$ to be rational where p and q are positive integers?

21-4. Decimals and Real Numbers

We have written our rational numbers as fractions. However there are advantages in using a decimal representation particularly when we wish to find which of two rational numbers is the larger. For example, it is difficult to decide by inspection whether $\frac{99}{70}$ or $\frac{140}{99}$ is the larger. But if we write the corresponding decimals

$$\frac{99}{70} = 1.4142857 \ldots$$

$$\frac{140}{99} = 1.414141 \ldots$$

we immediately see that the first number is larger than the second. To decide, it is sufficient to compare the first digits where the two decimals disagree (in the example, 2 and 1 in the fourth decimal place).

Each of these is an example of a non-terminating decimal. But rational numbers may have terminating decimals. For example,

$$\frac{1}{8} = .125$$

$$\frac{1}{5} = .2$$

The non-terminating decimals which represent rational numbers repeat, either from the beginning or after a certain number of digits. Thus

$$\frac{1}{3} = .3333 \ldots \qquad \frac{2}{11} = .181818 \ldots$$

$$\frac{1}{9} = .111111 \ldots \qquad \frac{1}{6} = .16666 \ldots$$

We usually save space by writing

$$\frac{1}{3} = .\overline{3}$$

$$\frac{2}{11} = .\overline{18}$$

$$\frac{1}{6} = .1\overline{6}$$

with a bar over the digit or succession of digits that repeats.

For a given repeating decimal it is easy to write the corresponding fraction. First we observe that

$$\frac{1}{9} = .1111\ldots = .\overline{1}$$

$$\frac{1}{99} = .0101\ldots = .\overline{01}$$

$$\frac{1}{999} = .001001\ldots = .\overline{001}.$$

Now suppose that the given decimal repeats from the beginning. We proceed as in the following example.

$$.171717\ldots = 17(.010101\ldots)$$
$$= 17 \times \frac{1}{99} = \frac{17}{99}$$

or more briefly

$$.\overline{17} = 17(.\overline{01}) = 17 \times \frac{1}{99} = \frac{17}{99}\ .$$

If the given decimal repeats only after a "bad start", we follow the method shown in this example.

$$.12333\ldots = .12\overline{3} = .12 + .00333\ldots = .12 + .00\overline{3}\ .$$

Since

$$.12 = \frac{12}{100} \quad \text{and} \quad .00\overline{3} = \frac{1}{100}(.\overline{3}) = \frac{1}{100}\cdot\frac{3}{9} = \frac{1}{300}\ ,$$

then

$$.12\overline{3} = \frac{12}{100} + \frac{1}{300} = \frac{37}{300}\ .$$

Does the decimal representation of $\sqrt{2}$ ultimately repeat? The answer is No. If it did repeat it would represent a rational number and we know that $\sqrt{2}$ is not a rational number.

We can summarize the facts about decimals as follows:

Rational Numbers: Terminating or repeating decimals

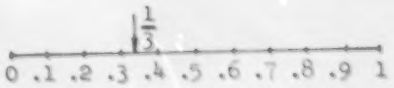
Irrational Numbers: Non-terminating, non-repeating decimals

What does it mean to say that a number is represented by a non-terminating decimal? For example, what does it mean to write

$$\frac{1}{3} = .333\ldots$$

where the digits 3 go on forever?

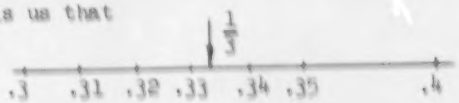
In the first place it means that $\frac{1}{3}$ is between .3 and .4, that is,



$$.3 < \frac{1}{3} < .4.$$

We now imagine that the interval [.3, .4] is subdivided into 10 parts, each of length .01. The decimal tells us that

$$.33 < \frac{1}{3} < .34.$$



If the interval [.33 , .34] is divided into 10 parts, each of length .001, our unending decimal tells us that

$$.333 < \frac{1}{3} < .334.$$

Continuing in this way we have the succession of statements

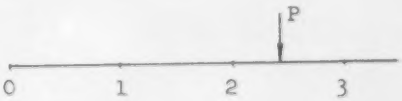
$$.3333 < \frac{1}{3} < .3334$$

$$.33333 < \frac{1}{3} < .33334$$

and so on.

The unending decimal .333 ... locates $\frac{1}{3}$ successively on intervals each of which lies within the preceding interval and is $\frac{1}{10}$ as long. We think of $\frac{1}{3}$ as locating the single point of intersection of all of these intervals. Thus every decimal locates exactly one point on the number line.

On the other hand, we assume that any point P on the number line has exactly one coordinate. We can see this as follows: P might be one of the points marked 0, 1, 2, 3, ... If not, it lies within an interval whose endpoints are marked by two successive integers; for example, the interval [2, 3].



Divide [2, 3] into the intervals [2, 2.1], [2.1, 2.2], [2.2, 2.3], ... to [2.9, 3]. P might be one of the points of division (like 2.3). If not, it lies inside one of the intervals. Continuing the subdivision we see that either one of two things must happen:

(1) P lies at a point of subdivision;

(2) P lies at no point of subdivision.

In the first case, P has a coordinate which is a terminating decimal; in the second, its coordinate is a non-terminating decimal. Numbers represented by decimals, whether terminating or non-terminating, are called real numbers.

To summarize, every point on the number line has a real number coordinate, and every real number is the coordinate of a point on the number line.

Exercises 21-4

1. Change the following fractions to decimals.

$$\frac{1}{20} \ , \ \frac{1}{30} \ , \ \frac{5}{6} \ , \ \frac{2}{7} \ .$$

2. Write the following decimals as fractions in lowest terms.

$$.2, \ .\overline{2}, \ .12, \ .\overline{12}, \ .321, \ .\overline{321}.$$

3. How can you tell without dividing whether a fraction $\frac{p}{q}$ is represented by a terminating or a non-terminating decimal?

4. What is the value of $.\overline{9} = .9999 \ldots$?

5. Consider .1010010001 ... where each time one more 0 is inserted between consecutive 1's. Does this represent a rational or an irrational number? Answer the same question for

.12345678910111213 ...

21-5. The Accuracy of the Divide and Average Method

In Chapter 20 we found that if a and b are any two positive numbers, the square of their average is greater than their product. That is

(1) $$(\frac{a+b}{2})^2 > ab.$$

We learned in fact that

(2) $$(\frac{a+b}{2})^2 = ab + (\frac{a-b}{2})^2.$$

This equation tells us by how much the square of the average exceeds the product ab.

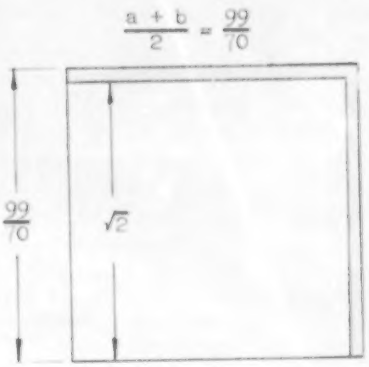
Let us apply these results to the problem of approximating $\sqrt{2}$ that we considered in Section 21-2. There we found two numbers $a = \frac{10}{7}$ and $b = \frac{7}{5}$ between which $\sqrt{2}$ lies. The product of a and b is 2. Their average, $\frac{a+b}{2}$, is $\frac{99}{70}$.

From (1)

$$(\frac{99}{70})^2 > 2$$

and therefore

$$\frac{99}{70} > \sqrt{2}.$$



A square $\frac{99}{70}$ on a side is larger than a square $\sqrt{2}$ on a side. If we knew the difference, $\frac{99}{70} - \sqrt{2}$, between the lengths of the sides of these two squares, we could find $\sqrt{2}$. We can think of this difference as the error e committed in using $\frac{99}{70}$ as an approximation to $\sqrt{2}$. We write therefore

$$e = \frac{99}{70} - \sqrt{2}.$$

Although we cannot immediately find this difference (e) between the sides of the squares, we do know exactly that the difference in their areas (area of larger square - area of smaller square) is

$$(\frac{99}{70})^2 - 2.$$

This is the area of the shaded region in the figure. In fact, from

(2) $$(\frac{a+b}{2})^2 = ab + (\frac{a-b}{2})^2$$

we have, replacing $\frac{a+b}{2}$ by $\frac{99}{70}$ and ab by 2,

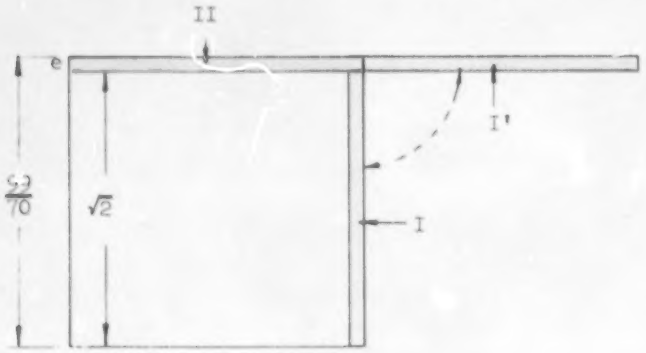
$$(\frac{99}{70})^2 = 2 + (\frac{a-b}{2})^2$$

and so

$$(\frac{22}{70})^2 - 2 = (\frac{a-b}{2})^2.$$

Since $a = \frac{10}{7}$ and $b = \frac{7}{5}$, $a - b = \frac{10}{7} - \frac{7}{5} = \frac{1}{35}$ and $(\frac{a-b}{2})^2 = (\frac{1}{70})^2 = \frac{1}{4900}$.

So the area of the shaded region (I and II) is <u>exactly</u> $\frac{1}{4900}$.



If we move the region I to I' so that it is in line with region II, we have a rectangle with the same area as the shaded region and the dimensions e and $(\frac{22}{70} + \sqrt{2})$.

Therefore

$$e \times (\frac{22}{70} + \sqrt{2}) = \frac{1}{4900}$$

and so

(3) $$e = \frac{1}{\frac{22}{70} + \sqrt{2}} \times \frac{1}{4900}.$$

Of course this answer for e contains the unknown $\sqrt{2}$. If in (3), we replace $\sqrt{2}$ by the larger number $\frac{22}{70}$, the denominator $\frac{22}{70} + \sqrt{2}$ becomes larger and so $\frac{1}{\frac{22}{70} + \sqrt{2}}$ becomes smaller. Therefore

$$e > \frac{1}{\frac{22}{70} + \frac{22}{70}} \times \frac{1}{4900}.$$

Since $\frac{22}{70}$ is close to $\sqrt{2}$, the expression on the right should be a rather good approximation to the actual error e.

Then

$$e \approx \frac{1}{\frac{22}{35}} \times \frac{1}{4900}$$

and

$$e \approx \frac{1}{99} \times \frac{1}{140} .$$

Let us find the right side in decimal form.

Since $\frac{1}{7} = .142857142857 \ldots$

$\frac{1}{2} \times \frac{1}{7} = \frac{1}{14} = .071428571428 57 \ldots$

and

$\frac{1}{10} \times \frac{1}{140} = .007142857142857 \ldots$

To divide by 99 we may divide by 9 and then by 11, obtaining first

$.00079365079365079 \ldots$

and then, finally,

$.00007215007215007 \ldots$

If we subtract this approximation to e from $\frac{22}{70}$ we should get a very good value for $\sqrt{2}$ (slightly too large. Why?).

Now $\frac{22}{70} = 1.4142857142857 \ldots$

Then $\sqrt{2} \approx 1.41428571428571428 \ldots$

$- \quad .00007215007215007 \ldots$

$1.41421356421356421 \ldots$

Actually $\sqrt{2} = 1.414213562373095 \ldots$

correct to 15 places, so that in fact our estimate gives correctly the first 8 decimal places! It is possible to show that our estimate has this degree of accuracy without knowing the value of $\sqrt{2}$.

Specifically it turns out that if E is the estimated value of the error e (.00007215 ...) the difference $(e - E) \approx \frac{E^2}{2m}$ where $m = 1.4142857 \ldots$ This works out to be about 2 in the 9^{th} decimal place.

Exercises 21-5

1. You learned in Section 21-2 (Exercises 2 and 3) that $\sqrt{3}$ is between $a = \frac{7}{4}$ and $b = \frac{12}{7}$. What is the product of a and b? What is their average m? Write m as a decimal to 10 places. Show that the error

$$e = m - \sqrt{3} > \frac{1}{97} \times \frac{1}{112}$$

$$= \frac{1}{97} \times \frac{1}{7} \times \frac{1}{16} .$$

Write the product of these three fractions as a decimal (to 10 places). Hint: Start with the decimal for $\frac{1}{16}$, divide by 7 and then by 97. Estimate $\sqrt{3}$ by subtracting your estimate for e from m. The correct value to 10 places is $\sqrt{3} = 1.7320508075$. How many of these places did you obtain correctly?

2. In Section 21-2 (Exercises 4 and 5) you found that

$$\frac{20}{9} < \sqrt{5} < \frac{9}{4} .$$

The average of $\frac{20}{9}$ and $\frac{9}{4}$ is $\frac{161}{72}$.

Change this fraction to a decimal (to 10 places).

Show that the error

$$e = \frac{161}{72} - \sqrt{5} > \frac{1}{161} \times \frac{1}{9} \times \frac{1}{16} .$$

Change the product on the right to a decimal (10 places) and subtract it from the decimal for $\frac{161}{72}$ (10 places). Compare your result with

2.2360680,

which is correct to 7 places.

21-6. The Square Root Function

Let us turn now to the properties of the square root function, that is, the function f which takes a number into its square root.

$$f : x \to \sqrt{x}.$$

What is the domain of f? That is, for what values of x is there a square root? Do all positive numbers have square roots? Now that we have introduced irrational numbers as well as rational numbers, we can answer Yes. Before this we could not sensibly talk even about $\sqrt{2}$.

Is 0 in the domain of f? Does $\sqrt{0}$ exist? Yes. In fact $\sqrt{0} = 0$ since $0^2 = 0 \times 0 = 0$.

Do negative numbers have square roots? For example, can we give any meaning to the square root of -1? Certainly not, using the numbers that we know about.* For, let

$$\sqrt{-1} = a$$

where a is any real number, positive, negative or 0. Then

$$a^2 = -1.$$

But the square of any real number is positive or 0. There is no point on the number line whose coordinate can be squared to give -1 or any other negative number.

We can now say that the domain of the square root function f is the set of all non-negative real numbers $\{x : x \geq 0\}$.

What is the range of f. That is, what is the set of all possible numbers which can be square roots? Here we must be careful because every positive number has two square roots. For example, 4 has the square root 2 and also the square root -2. In fact $2^2 = 4$ and $(-2)^2 = 4$.

Of these two square roots, the positive one is the more important for most purposes. The symbol $\sqrt{\ }$ is used to name this one so that we write

$$\sqrt{4} = 2$$

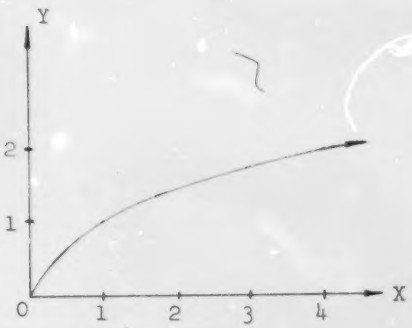
but not $\sqrt{4} = -2$. This is an agreement about the way to use the symbol

---

* As you go on in mathematics you will learn about a new kind of number. For this new kind of number it is possible to give a meaning to $\sqrt{-1}$.

$\sqrt{\ }$. It is a matter of convenience. Since we do not wish to be in any doubt about the meaning of the symbols that we use, we agree that $\sqrt{4}$, for example, denotes a particular choice between the two possible square roots (the non-negative one). With this understanding, $\sqrt{x} \geq 0$. The range of $f : x \to \sqrt{x}$ can include no negative numbers. It does include 0 and all positive numbers b, because to say that b is the square root of a number x, that is, that $b = \sqrt{x}$, is to say that $x = b^2$. Given any b, the x of which it is the square root, is found by squaring b.

What does the graph of $y = \sqrt{x}$ look like? We can easily locate the points $(0, 0)$, $(1, 1)$, $(2, \sqrt{2})$, $(3, \sqrt{3})$ and $(4, 2)$. By drawing a smooth curve through these points we get a figure like this. Note that there are no points to the left of $x = 0$ and none below the X-axis. Why not?



It looks as if the graph always rises as we go from left to right. This means that if we choose two numbers a and b with

$$a < b$$

then

$$\sqrt{a} < \sqrt{b}.$$

(Of course, we must have $a \geq 0$. Why?)

Is this necessarily true? The answer is Yes. First we will prove: If $\sqrt{b} < \sqrt{a}$, then $b < a$. This result will be needed.

If $\sqrt{b} < \sqrt{a}$,

then $\sqrt{b} \cdot \sqrt{b} < \sqrt{a}\ \sqrt{b}$ and $\sqrt{a} \cdot \sqrt{b} < \sqrt{a} \cdot \sqrt{a}$. (Multiplication Property of Order)

Hence $\sqrt{b} \cdot \sqrt{b} < \sqrt{a} \cdot \sqrt{a}$ (Transitive Property of Order)

or $(\sqrt{b})^2 < (\sqrt{a})^2$.

Therefore $b < a$ (Definition of square root)

Now if $\sqrt{a}$ were not less than $\sqrt{b}$ when $a < b$, then either

$$\sqrt{a} = \sqrt{b}$$

or $$\sqrt{b} < \sqrt{a}.$$

In the first case we would conclude that $a = b$; in the second case, that $b < a$. Since we were given that $a < b$, both of these conclusions are false. Therefore, the correct conclusion is that when $a < b$, $\sqrt{a} < \sqrt{b}$.

We can now say with assurance that the function

$$f : x \to \sqrt{x}$$

is an <u>increasing function</u> over its whole domain $(x : x \geq 0)$.

It also appears from our figure that the graph becomes less steep as we go from left to right. Consider the effect of an increase of x by 1. If x increases from 0 to 1, $\sqrt{x}$ increases from 0 to 1. Now if x increases from 1 to 2, $\sqrt{x}$ increases from 1 to $\sqrt{2}$, that is by $\sqrt{2} - 1 \approx .41$. From $x = 49$ to $x = 50$, $\sqrt{x}$ increases only $\sqrt{50} - \sqrt{49} \approx 7.07 - 7 = .07$. Let us go out on our graph, to larger values of x. How much is $\sqrt{10,001} - \sqrt{10,000}$? How much is $\sqrt{1,000,001} - \sqrt{1,000,000}$? The divide and average method will tell you. In the exercises you will be asked to show that the answers are approximately .005 and .0005. So we see that in fact the graph becomes less and less steep as x increases.

<u>Exercises 21-6a</u>

1. Estimate $\sqrt{10,001}$ by the divide and average method, and verify that

$$\sqrt{10,001} - \sqrt{10,000} \approx .005.$$

2. Estimate $\sqrt{1,000,001}$ by the divide and average method, and verify that

$$\sqrt{1,000,001} - \sqrt{1,000,000} \approx .0005.$$

3. Show that $\sqrt{a+1} - \sqrt{a} = \dfrac{1}{\sqrt{a+1} + \sqrt{a}} < \dfrac{1}{2\sqrt{a}}$. Use this result with $a = 10,000$ and $a = 1,000,000$. Compare the result with Exercise 1 and Exercise 2.

4. From the result of Exercise 3, show that the increase $(\sqrt{a+1} - \sqrt{a})$ can be made as small as you please by choosing a sufficiently large value of <u>a</u>. For example, make $\sqrt{a+1} - \sqrt{a}$ less than $\dfrac{1}{1,000,000}$ by choosing <u>a</u> large enough.

What is the connection between multiplying and taking the square root? In particular, is it true that

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}\ ?$$

Let us experiment.

Exercise 21-6b

(Class Discussion)

1. Test the truth of $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ for

$a = 1, \quad b = 1$

$a = 0, \quad b = 1$

$a = 1, \quad b = 2$

$a = 4, \quad b = 9$

Our tests are favorable, but of course they do not prove that the statement is true for all non-negative values of a and b. Even with the restriction of a and b to whole numbers the claim that

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

is true includes an infinite number of assertions! No finite set of examples could establish this claim. What we need is a general proof.

To get an idea of how such a general proof could be given, consider a more difficult special case than we have had so far.

Is $\sqrt{2 \times 3} = \sqrt{2} \times \sqrt{3}$? To answer this, we could calculate $\sqrt{6}$, $\sqrt{2}$ and $\sqrt{3}$ to several decimal places to see whether the results seem to agree. This is <u>not</u> a good idea. Why not?

In the first place, all that we could show in this way is that $\sqrt{2 \times 3}$ and $\sqrt{2} \times \sqrt{3}$ are <u>approximately</u> equal. We would not be sure that they are <u>exactly</u> equal. More importantly this proof would give us no clue about how to proceed in the general case. We cannot hope in this way to check our statement for <u>all</u> a and b.

Let us fall back on the <u>idea</u> of a square root, that is on what we <u>mean</u> by the square root of a number. To say that some number is $\sqrt{2 \times 3}$ is to say that the square of this number is 2×3. If then it is true that

$$\sqrt{2 \times 3} = \sqrt{2} \times \sqrt{3},$$

then it must be true that

$$(\sqrt{2} \times \sqrt{3})^2 = 2 \times 3.$$

Is it? How do we square a product? By squaring each factor and multiplying.

So $(\sqrt{2} \times \sqrt{3})^2 = (\sqrt{2})^2 \times (\sqrt{3})^2$. But $(\sqrt{2})^2 = 2$ and $(\sqrt{3})^2 = 3$, again using the idea of what a square root means. So our proof has a happy ending.

Now we can generalize to any non-negative a and b. Is $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$? Yes, if and only if

$$(\sqrt{a} \times \sqrt{b})^2 = a \times b.$$

But $(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a})^2 \times (\sqrt{b})^2$ since the square of the product of any two numbers is the product of their squares.

Now we are almost done. $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$ from the very meaning of square root. Hence

$$(\sqrt{a} \times \sqrt{b})^2 = a \times b$$

as required.

In the next section we shall find how very useful this result is.

Exercises 21-6c

1. Show that $\sqrt{2 \times 5} = \sqrt{2} \times \sqrt{5}$ patterning your argument on that given for

$$\sqrt{2 \times 3} = \sqrt{2} \times \sqrt{3}.$$

2. Show by examples that it is not always true that $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$.

3. Show that $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ if $a = 0$ or $b = 0$. Now show that $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ is true only if $a = 0$ or $b = 0$ (or both).

   Hint: Square both sides. Note that

$$(\sqrt{a} + \sqrt{b})^2 = a + 2(\sqrt{a} \times \sqrt{b}) + b.$$

4. Show by a counter-example that it is not always true that

$$\sqrt{a - b} = \sqrt{a} - \sqrt{b}.$$

   Could it ever be true?

5. Is it true that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for all positive a and b?

   Test the statement with particular numbers. Then construct a general proof following the example of the one in the text.

21-7. Simplifying Radicals

If we were required to find the square roots of the integers from 1 to 100, we would notice that for the perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, the square roots are integers which can be written at once. In other cases, we have the divide and average method. Nice as this method is, it does take time; and it would be pleasant if there were a quicker way. Actually, in many cases, we can use the fact that

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

to simplify the computation.

For example, we can write

$$\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3}.$$

Therefore if we have already computed

$$\sqrt{3} = 1.732 \ldots$$

we know at once that

$$\sqrt{12} = 2 \times (1.732 \ldots) = 3.464 \ldots$$

Similarly,

$$\begin{aligned} \sqrt{50} &= \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} \\ &= 5\sqrt{2} \\ &= 5(1.414 \ldots) \\ &= 7.070 \ldots \end{aligned}$$

(Actually $\sqrt{50} = 7.071 \ldots$)

In these examples, it was possible to write the number under the radical sign ($\sqrt{\ }$) as the product of two integers, one of which is a perfect square.

Exercises 21-7a

1. How many integers from 1 to 100 inclusive are either perfect squares or contain perfect squares as factors? List these integers omitting the 10 perfect squares given in the text. How many square roots remain to be computed?

2. Suppose that we had a list of the values of $\sqrt{n}$ for all integers n from 1 to 100. How could we find a list of values for $\sqrt{100n}$?

The theorem $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ can be used in other cases to reduce the amount of arithmetic. Since

$$\sqrt{6} = \sqrt{2} \times \sqrt{3},$$

if we already know $\sqrt{2}$ and $\sqrt{3}$, we can multiply them to obtain a decimal approximation to $\sqrt{6}$. If you did Exercise 1 correctly, you found that in the list of values of $\sqrt{n}$ for $n = 1, 2, \ldots, 100$ there remain 60 square roots to be calculated. Actually we need only the square roots of the <u>prime numbers</u>. All other square roots can be found from this list by multiplying. For example,

$$\sqrt{21} = \sqrt{3} \times \sqrt{7}$$

$$\sqrt{30} = \sqrt{2} \times \sqrt{3} \times \sqrt{5}.$$

There are 24 primes less than 100 whose square roots need to be computed.

The following table includes the square roots, to 3 decimal places, of the integers from 1 to 100. Hereafter you may refer to this table for square roots that are needed.

SQUARE ROOTS OF INTEGERS FROM 1 TO 100 TO THREE DECIMAL PLACES

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
1	1.000	26	5.099	51	7.141	76	8.718
2	1.414	27	5.196	52	7.211	77	8.775
3	1.732	28	5.292	53	7.280	78	8.832
4	2.000	29	5.385	54	7.348	79	8.888
5	2.236	30	5.477	55	7.416	80	8.944
6	2.449	31	5.568	56	7.483	81	9.000
7	2.646	32	5.657	57	7.550	82	9.055
8	2.828	33	5.745	58	7.616	83	9.110
9	3.000	34	5.831	59	7.681	84	9.165
10	3.162	35	5.916	60	7.746	85	9.220
11	3.317	36	6.000	61	7.810	86	9.274
12	3.464	37	6.083	62	7.874	87	9.327
13	3.606	38	6.164	63	7.937	88	9.381
14	3.742	39	6.245	64	8.000	89	9.434
15	3.873	40	6.325	65	8.062	90	9.487
16	4.000	41	6.403	66	8.124	91	9.539
17	4.123	42	6.481	67	8.185	92	9.592
18	4.243	43	6.557	68	8.246	93	9.644
19	4.359	44	6.633	69	8.307	94	9.695
20	4.472	45	6.708	70	8.367	95	9.747
21	4.583	46	6.782	71	8.426	96	9.798
22	4.690	47	6.856	72	8.485	97	9.849
23	4.796	48	6.928	73	8.544	98	9.899
24	4.899	49	7.000	74	8.602	99	9.950
25	5.000	50	7.071	75	8.660	100	10.000

Exercises 21-7b

In each of the following express the number under the radical sign as the product of an integer between 1 and 100 and a perfect square. Then calculate the square root using the table in the text. For example, $\sqrt{297} = \sqrt{9 \times 33} = 3\sqrt{33} \approx 3 \times 5.745 = 17.235$.

1. $\sqrt{8000}$
2. $\sqrt{152}$
3. $\sqrt{128}$
4. $\sqrt{680000}$
5. $\sqrt{640}$
6. $\sqrt{(35)^3}$

Use the fact that for all positive numbers a and b $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ to calculate the following square roots. Examples:

$$\sqrt{\frac{83}{4}} = \frac{\sqrt{83}}{\sqrt{4}} = \frac{\sqrt{83}}{2} \approx \frac{9.110}{2} = 4.555.$$

$$\sqrt{\frac{5}{7}} = \sqrt{\frac{5 \times 7}{49}} = \frac{\sqrt{35}}{7} \approx \frac{5.916}{7} = .845.$$

7. $\sqrt{\frac{41}{36}}$
8. $\sqrt{\frac{13}{3}}$
9. $\sqrt{.68}$
10. $\sqrt{.2}$
11. $\sqrt{.001}$
12. $\sqrt{(-.36)^2}$

Suppose that we wish to find $\sqrt{34.2}$. We may use our divide and average method. Beginning with 6 as a first approximation (too large), we compute $\frac{34.2}{6} = 5.7$ and average to obtain 5.85. We could, of course, repeat this process to get greater accuracy. However, there is an alternative

method. We know from the table that

$$\sqrt{34} = 5.831$$

$$\sqrt{35} = 5.916$$

The increase of n from 34 to 35 is 1. The corresponding increase in $\sqrt{n}$ is

$$5.916 - 5.831 = .085.$$

34.2 is .2 or $\frac{1}{5}$ of the way from 34 to 35. Let us assume that the required result for $\sqrt{34.2}$ is about $\frac{1}{5}$ of the way from 5.831 to 5.916. This gives

$$5.831 + \frac{1}{5}(.085) = 5.831 + .017 = 5.848.$$

(The correct value of $\sqrt{34.2} = 5.848077\ldots$ so actually we are fairly close.) This method, called interpolation, is based on the idea that over a short x-interval the graph of $y = \sqrt{x}$ is nearly straight.

Exercises 21-7c

1. Compute $\sqrt{83.7}$ by two methods; divide and average (once) starting with 9 as the first estimate, and interpolation. Compare the two results.

2. If you use the divide and average method is the result too large or too small?

   Can you show that the interpolation method gives a result which is always too small?

   Hint: Is the graph of $y = \sqrt{x}$ above or below a line segment (chord) joining two of its points?

21-8. Some Problems

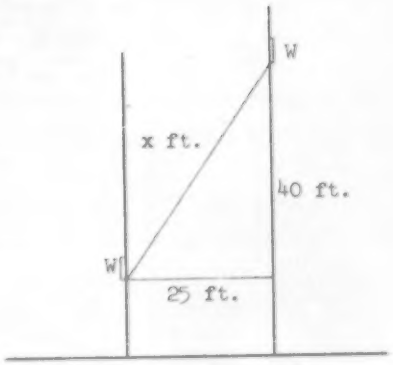
We started this chapter by asking how far it is to the horizon from a point on top of a mountain. We answered this question by finding one side of a right triangle whose other two sides are known. We used the Pythagorean property of right triangles, and were led to consider how to find square roots. Now that we know how to do this, we are prepared to solve many other problems which lead to square roots. Most of these problems arise from the need to

find the hypotenuse or one leg of a right triangle.

We give some examples:

Example 1

On a street 25 feet wide, a window on one side of the street is 40 feet above a window across the street. In case one, how long a ladder would it take to reach from one window to the other?



It is clear from the figure that we must find the value of x where

$$x^2 = 25^2 + 40^2 = 2225.$$

So

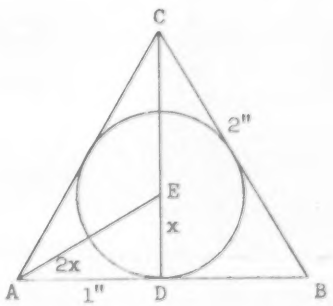
$$x = \sqrt{2225} = 5\sqrt{89}.$$

Using the table we find that

$$x \approx 5 \times 9.43 = 47.15.$$

Example 2

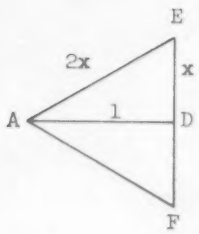
A circular cylinder is to be fitted tightly into a hole which is bounded by an equilateral triangle with sides 2 inches long. What must its radius be?



Solution: The figure shows a cross-section of the hole and the cylinder. A perpendicular from E, the center of the circle, bisects the base AB. The triangle ADE is a right triangle which is half of another equilateral triangle AEF.

Hence $AE = 2DE = 2x$.

Then applying the Pythagorean property to triangle ADE



$$(2x)^2 = x^2 + 1^2$$

$$4x^2 = x^2 + 1$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \sqrt{\frac{1}{3}} \approx \sqrt{.33} = \frac{1}{10}\sqrt{33} \approx .574$$

or, more accurately

$$x = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{3} \approx \frac{1.732}{3} = .577^{+}.$$

Exercises 21-8

1. The students attending Lincoln High School have a habit of cutting across a vacant lot near the school instead of following the sidewalk around the corner. The lot is a rectangular lot 200 feet by 300 feet, and the short-cut follows a straight line from one corner of the lot to the opposite corner. How long, to the nearest foot, is the short-cut? How much walking do they save by taking the short-cut? If the ground is so rough that it takes 25 % longer to walk 100 feet of path than 100 feet of sidewalk, do they save any time? If so, how much?

2. A fourteen foot ladder rests against a vertical wall, the foot of the ladder being seven feet from the base of the wall. Determine the height at which the ladder touches the wall. How close an approximation would be reasonable to expect in this case?

3. Two roads cross each other at right angles. Two cars are at the cross-roads at noon. One is going at 50 miles per hour, the other at 65 miles per hour. If each car maintains its speed, how far are they apart (air line) at 2 p.m.?

4. In the previous problem assume that the first car is at the cross-roads at noon and the other car reaches the cross-roads at 1 p.m. How far apart are the cars at 2 p.m.?

5. A baseball diamond is a square 90 feet on a side. The pitcher's mound is about 60 feet from home plate on the line joining home plate and second base.

   (a) How far is it from home plate to second base?

   (b) How far is it from the pitcher's mound to first base?

   (c) If the pitcher has a choice whether to throw to first, second or third, which base should he choose?

6. Find the altitude of an isosceles triangle with base 12 feet and equal sides of length 17 feet.

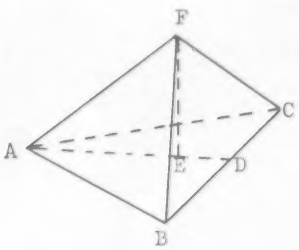
7. A regular hexagon (6 sides) is inscribed in a circle of radius 5 feet. What is the perpendicular distance from the center to any side?

8. A regular hexagon is <u>circumscribed</u> about a circle of radius 5 feet. What is the perimeter of the hexagon?

9. A certain pyramid has 4 faces, each of which is an equilateral triangle of side 2 inches. Find the altitude of the pyramid.

   <u>Hint</u>: In the figure, what kind of a triangle is AEF?

   What is AE?



10. A ball is dropped from a height of 95 feet. How long does it take to hit the ground if the distance x feet, fallen in t seconds, is given by $x = 16t^2$?

21-9. <u>Summary</u>

It is proved that the square root of a positive integer n cannot be a rational number unless n is a perfect square. We can however use a rational number <u>a</u> to approximate $\sqrt{n}$ and then bracket $\sqrt{n}$ between <u>a</u> and $b = \frac{n}{a}$. A better estimate is obtained by averaging <u>a</u> and <u>b</u>.

This average, $m = \frac{a + b}{2}$, is always too large unless $a = b$ which is possible only if n is a perfect square.

By using m as a new estimate and dividing n by m, we bracket $\sqrt{n}$ between m and $\frac{n}{m}$. Averaging again and continuing the process we can approximate $\sqrt{n}$ to any desired accuracy.

Instead of repeating the divide and average process, we can use the estimate $\frac{1}{2m}(\frac{a - b}{2})^2$ for the error made in replacing $\sqrt{n}$ by the average m of a and b. If we subtract this estimate from m we obtain a very accurate estimate for $\sqrt{n}$ when $\frac{a - b}{2}$ is small (say .02 or less).

The domain of the function

$$f : x \to \sqrt{x}$$

is the set of non-negative real numbers. The range is the same set... $\sqrt{x}$ increases for all x in the domain but the rate of increase becomes less as x is increased. Calculation of square roots is greatly simplified by using the fact that for all positive numbers a and b,

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

and

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} .$$

We review briefly the meaning of non-ending decimals and recall that the decimal representations of

(1) rational numbers either terminate or repeat,

(2) irrational numbers neither terminate nor repeat.

The set of real numbers includes the set of rational numbers and the set of irrational numbers.